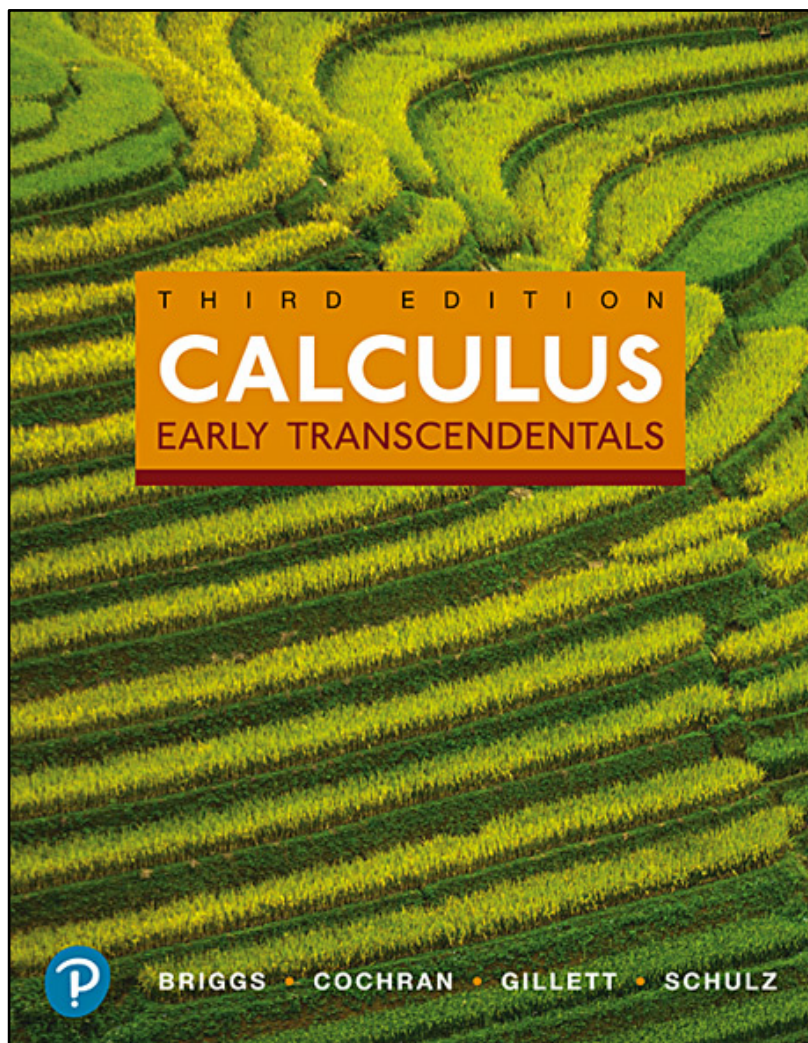


Calculus Early Transcendentals

Third Edition



- optimization
- linear approximation

Chapter 4

Applications of the Derivative

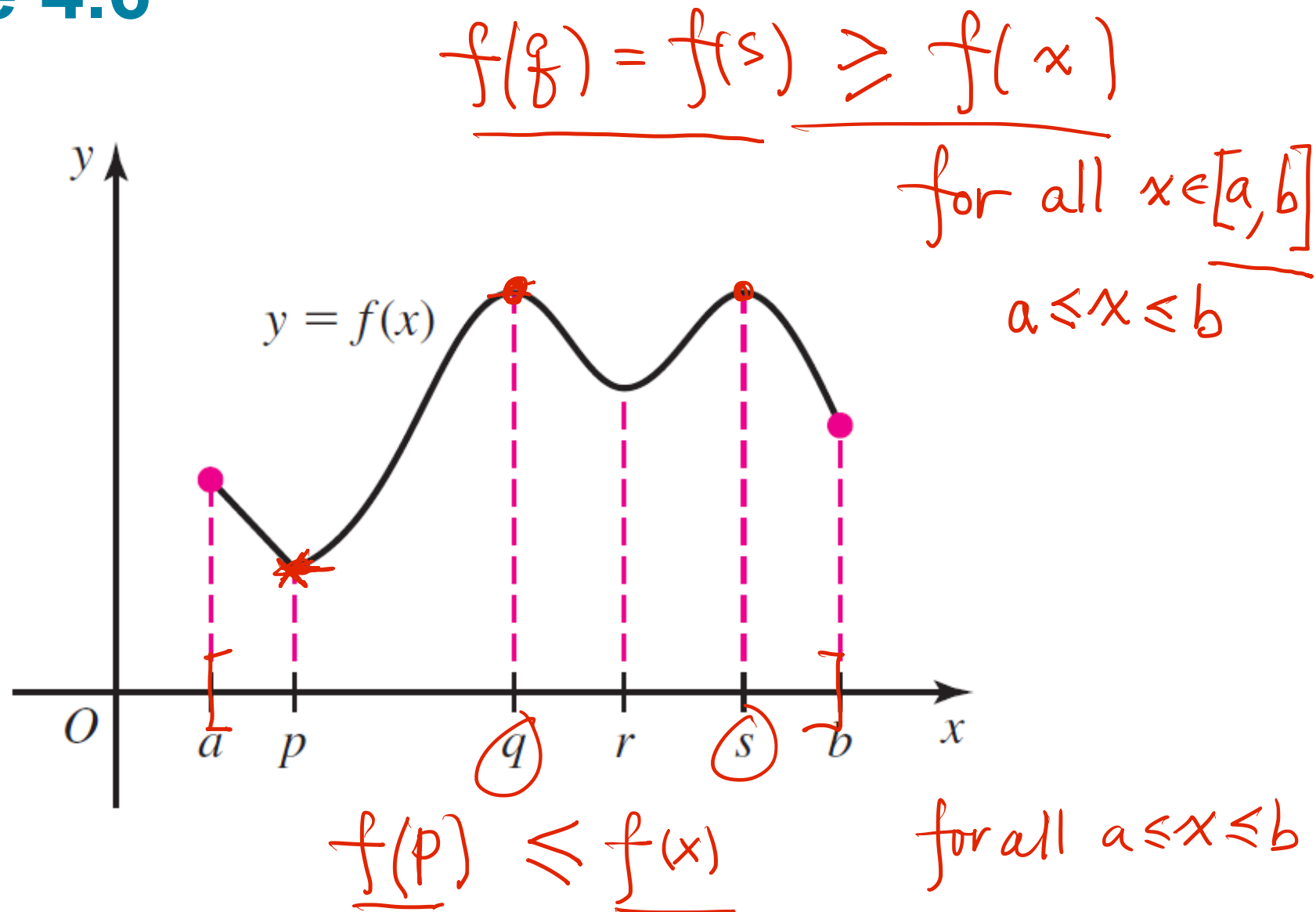
9 lectures

4.1, 4.2-3, 4.3, 4.4, 4.5a, 4.5b, 4.6, 4.7, 4.9

Section 4.1 Maxima and Minima

HW #5, 9, 11, 14, 15, 19, 31, 33, 35, 41, 43, 49, 53, 55, 89,

Figure 4.6



Definition Absolute Maximum and Minimum

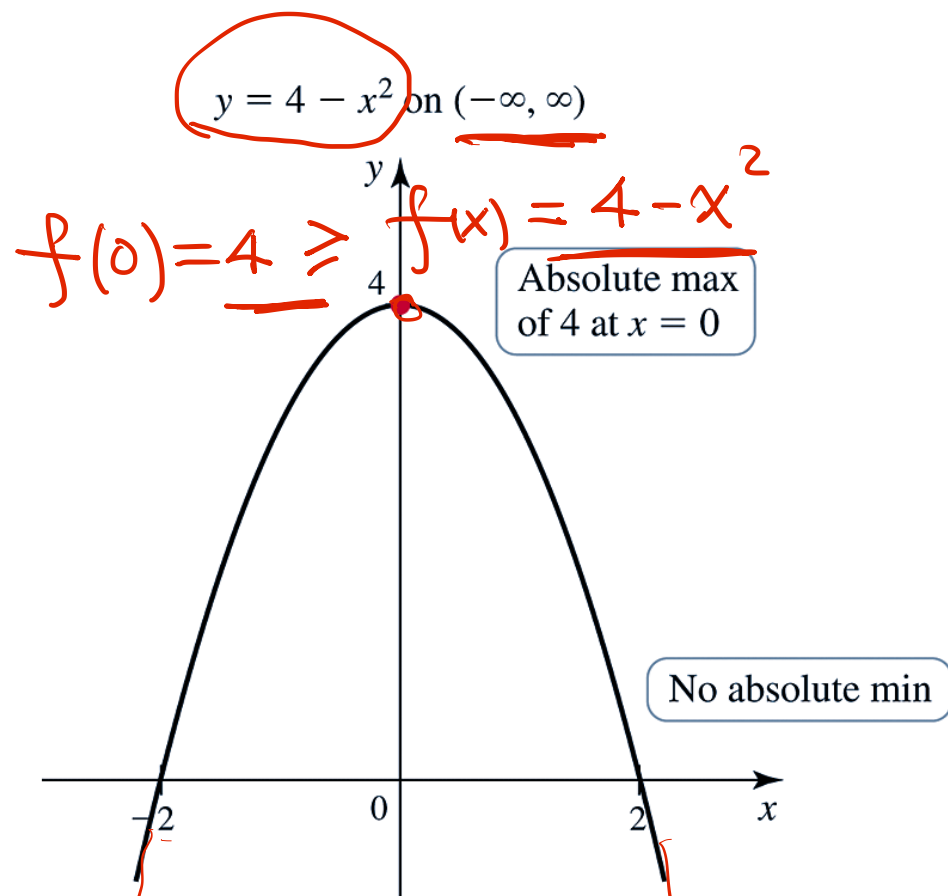
Let f be defined on a set D containing c . If $f(c) \geq f(x)$ for every x in D , then $f(c)$ is an **absolute maximum** value of f on D . If $f(c) \leq f(x)$ for every x in D , then $f(c)$ is an **absolute minimum** value of f on D .

An **absolute extreme value** is either an absolute maximum or an absolute minimum value.

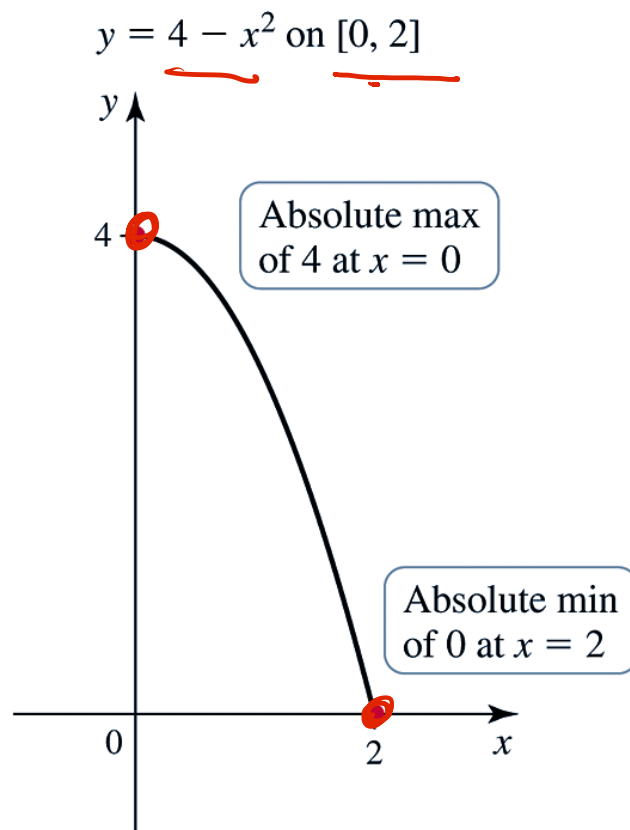
Theorem 4.1 Extreme Value Theorem

A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

Figure 4.2 (a & b)



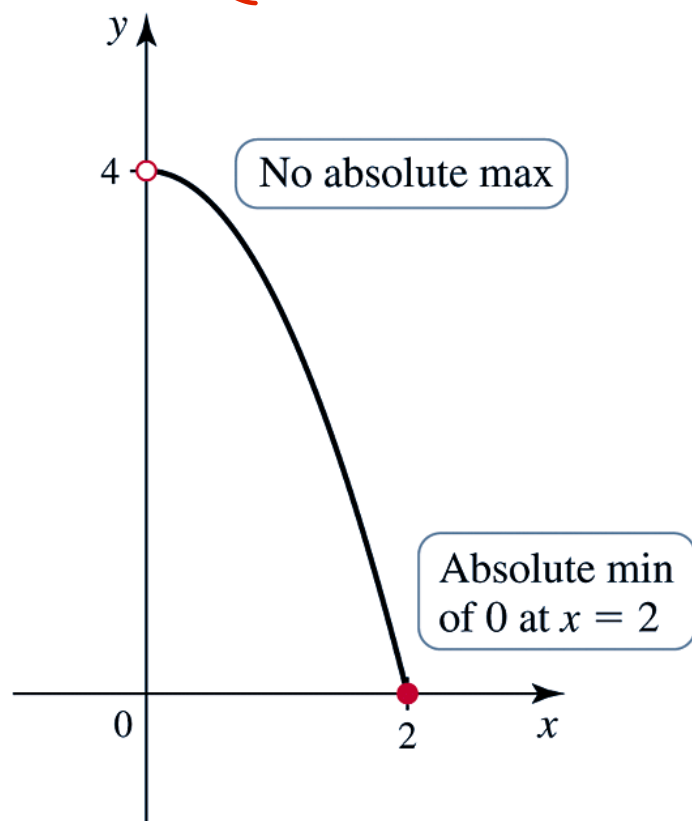
(a) Absolute max only



(b) Absolute max and min

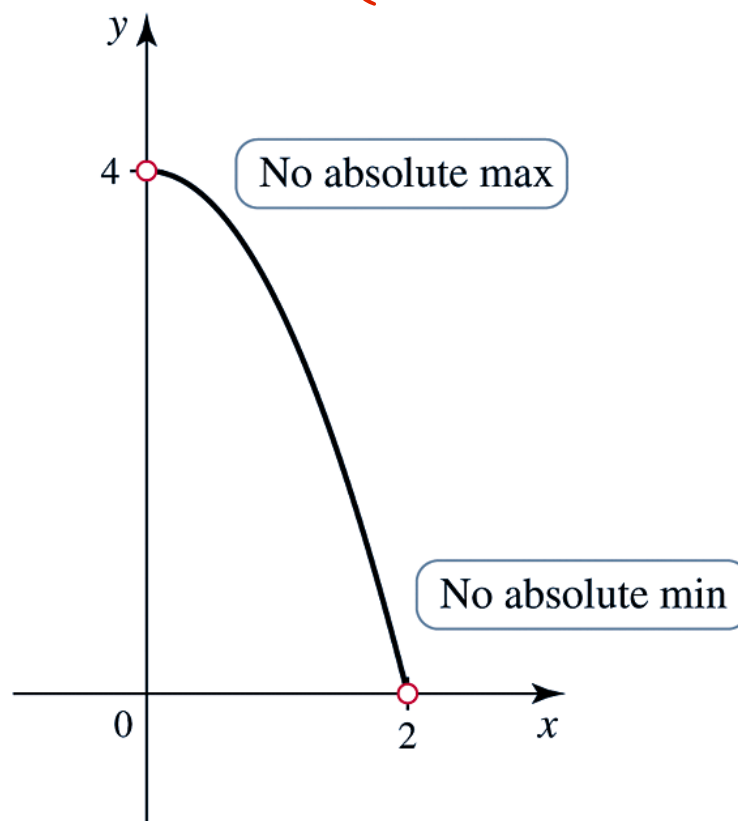
Figure 4.2 (c & d)

$$y = 4 - x^2 \text{ on } (0, 2]$$



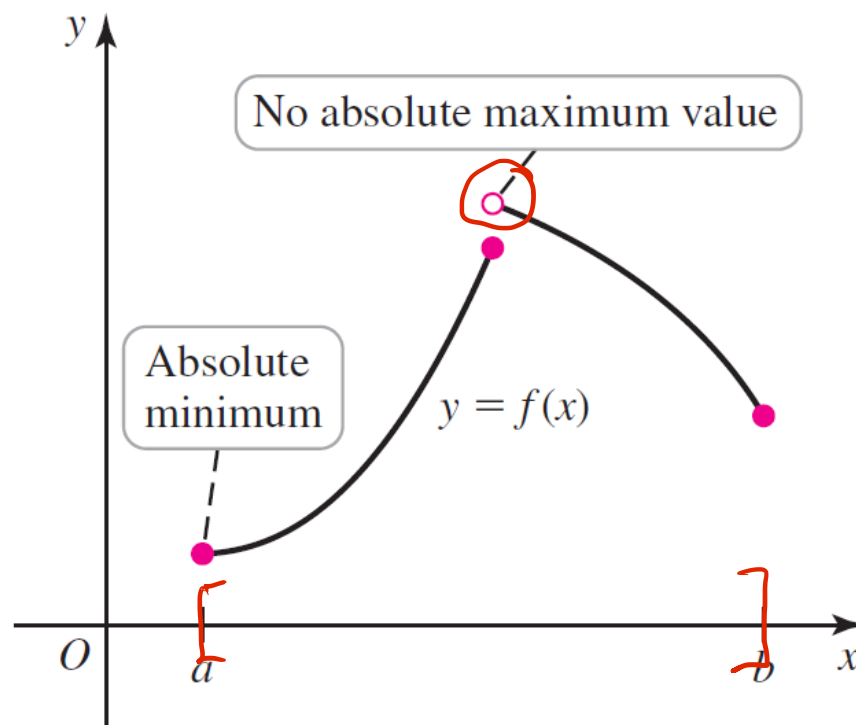
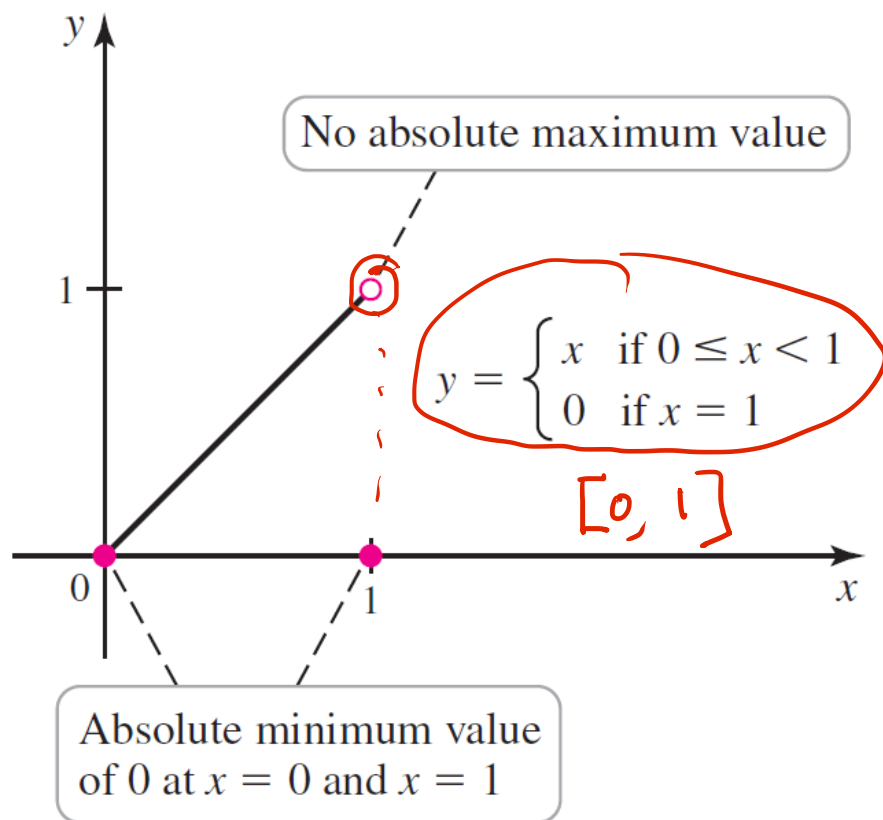
(c) Absolute min only

$$y = 4 - x^2 \text{ on } (0, 2)$$



(d) No absolute max or min

Figure 4.3 (a & b)



Example 1

Figure 4.4 (a & b)

$[a, b]$ & cont.

discont. $\Rightarrow$ no ab. min.

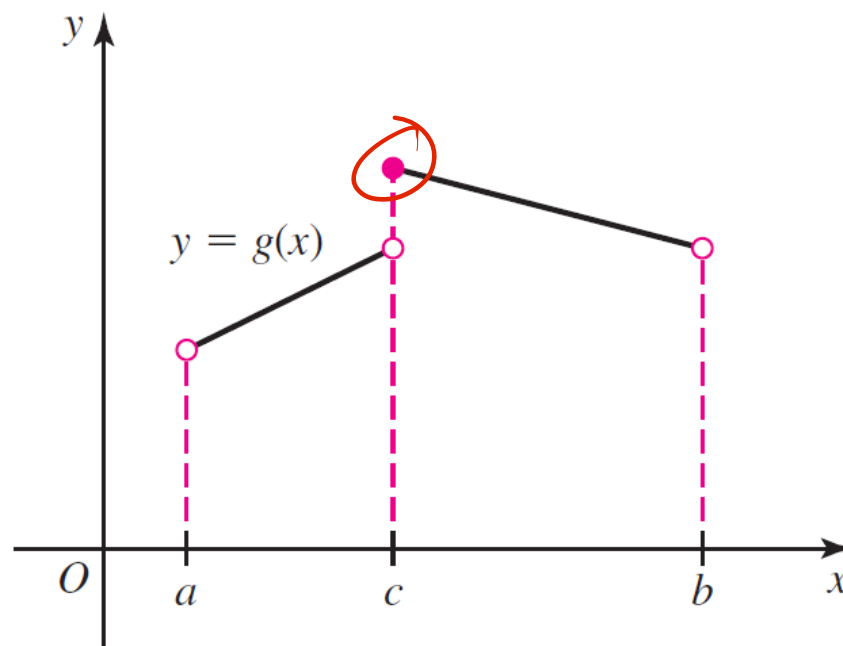
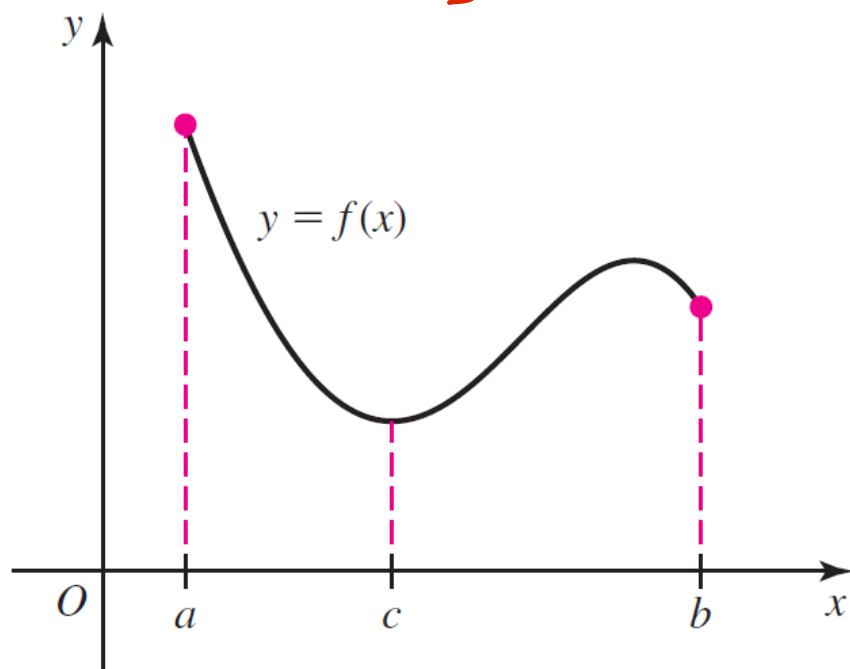
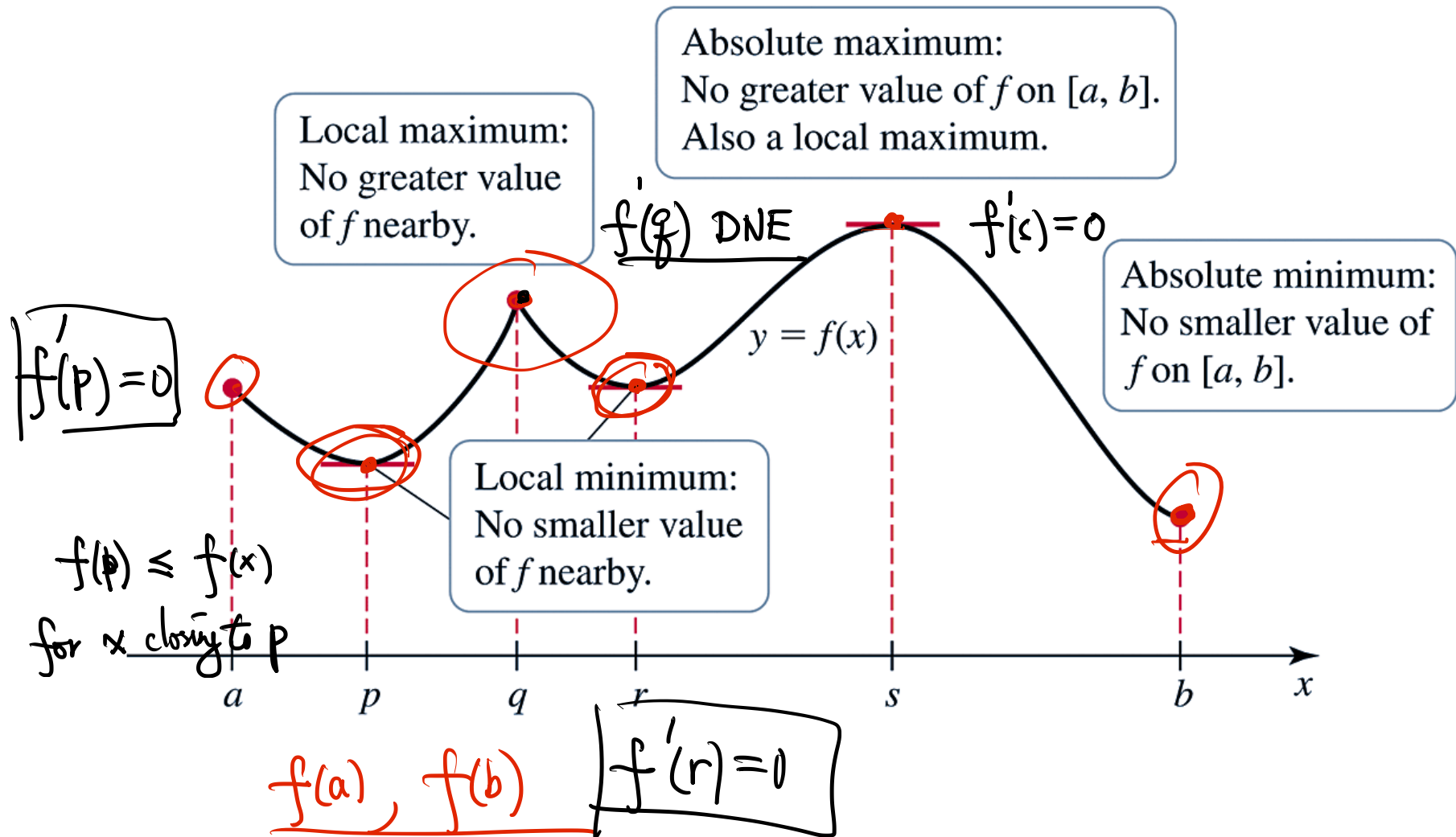


Figure 4.5



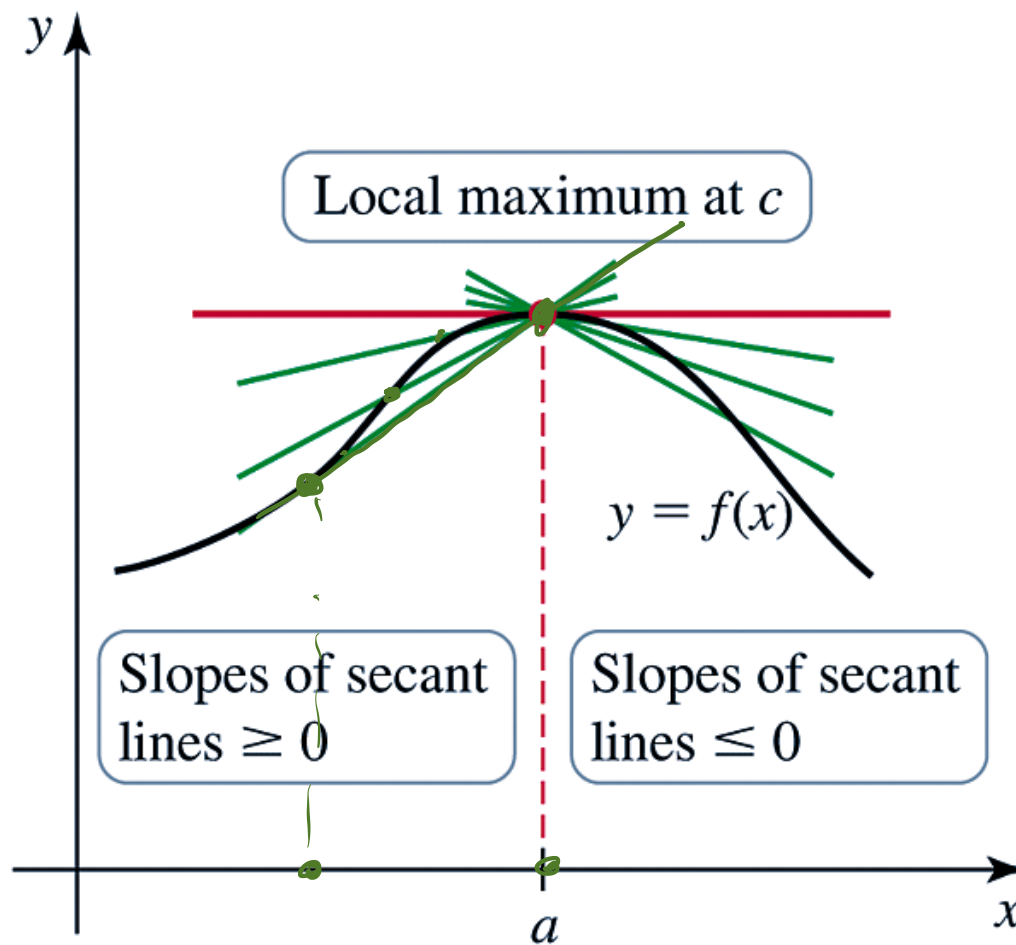
Definition Local Maximum and Minimum Values

Suppose c is an interior point of some interval I on which f is defined. If

$f(c) \geq f(x)$ for all x in I , then $f(c)$ is a **local maximum** value of f . If

$f(c) \leq f(x)$ for all x in I , then $f(c)$ is a **local minimum** value of f .

Figure 4.7



Theorem 4.2 Local Extreme Value Theorem

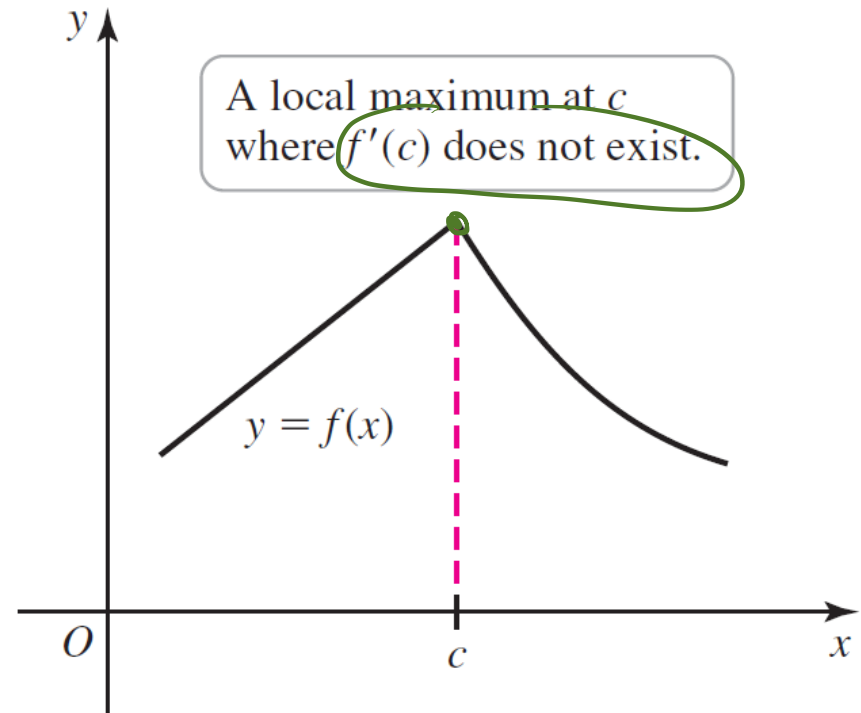
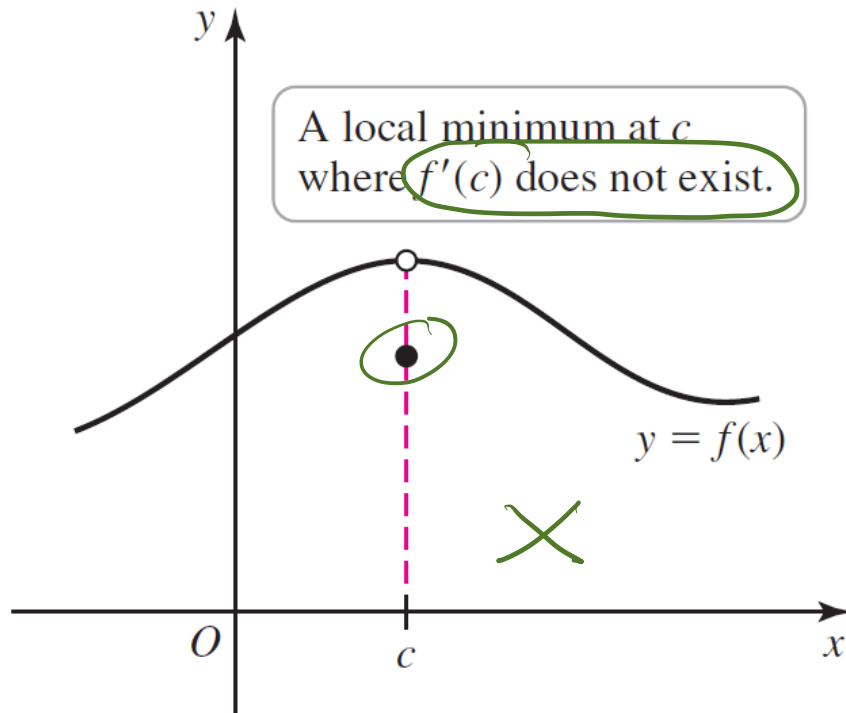
If f has a local maximum or minimum value at c and $f'(c)$ exists, then $f'(c) = 0$ 1st-der. test

Definition Critical Point

An interior point c of the domain of f at which $f'(c) = 0$ or $f'(c)$ fails to exist is called a **critical point** of f .

critical points $\implies$ local extreme values

Figure 4.8 (a & b)



critical points $\nRightarrow$ local extreme values

Figure 4.9 (a)

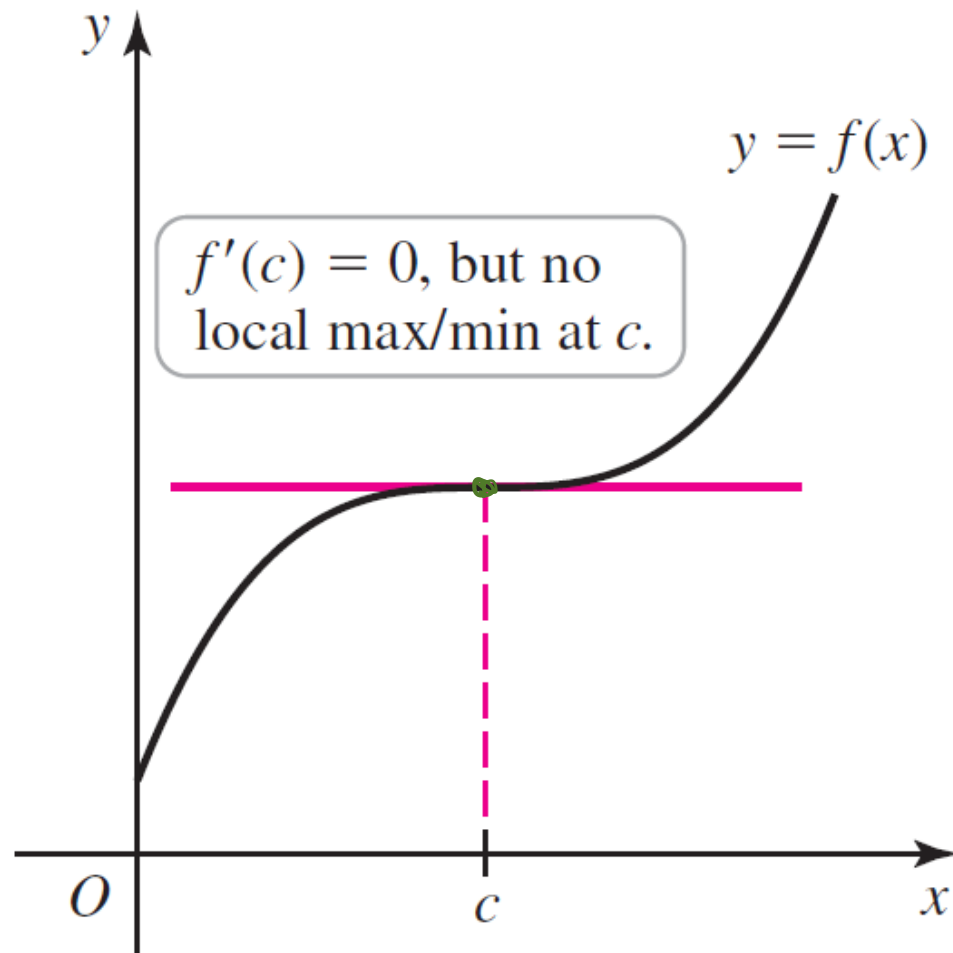
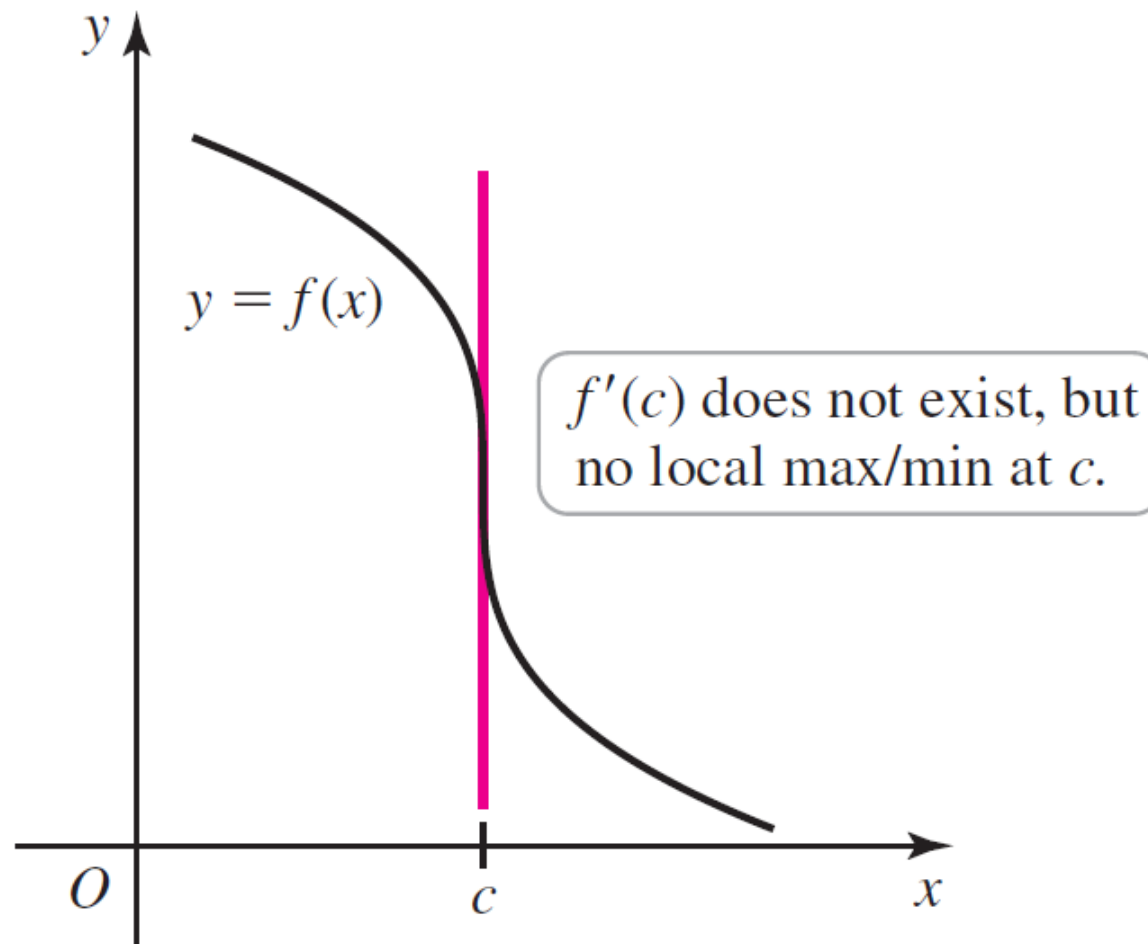


Figure 4.9 (b)



Example 3 Find critical points of $f(x) = x^2 \ln x$.

1st step

$$0 = f'(x) = (x^2)' \ln x + x^2 (\ln x)'$$

$$= 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x = 0$$

2nd step solve this eq.

$$y = x^2 \ln x \quad 0 = x [2 \ln x + 1]$$

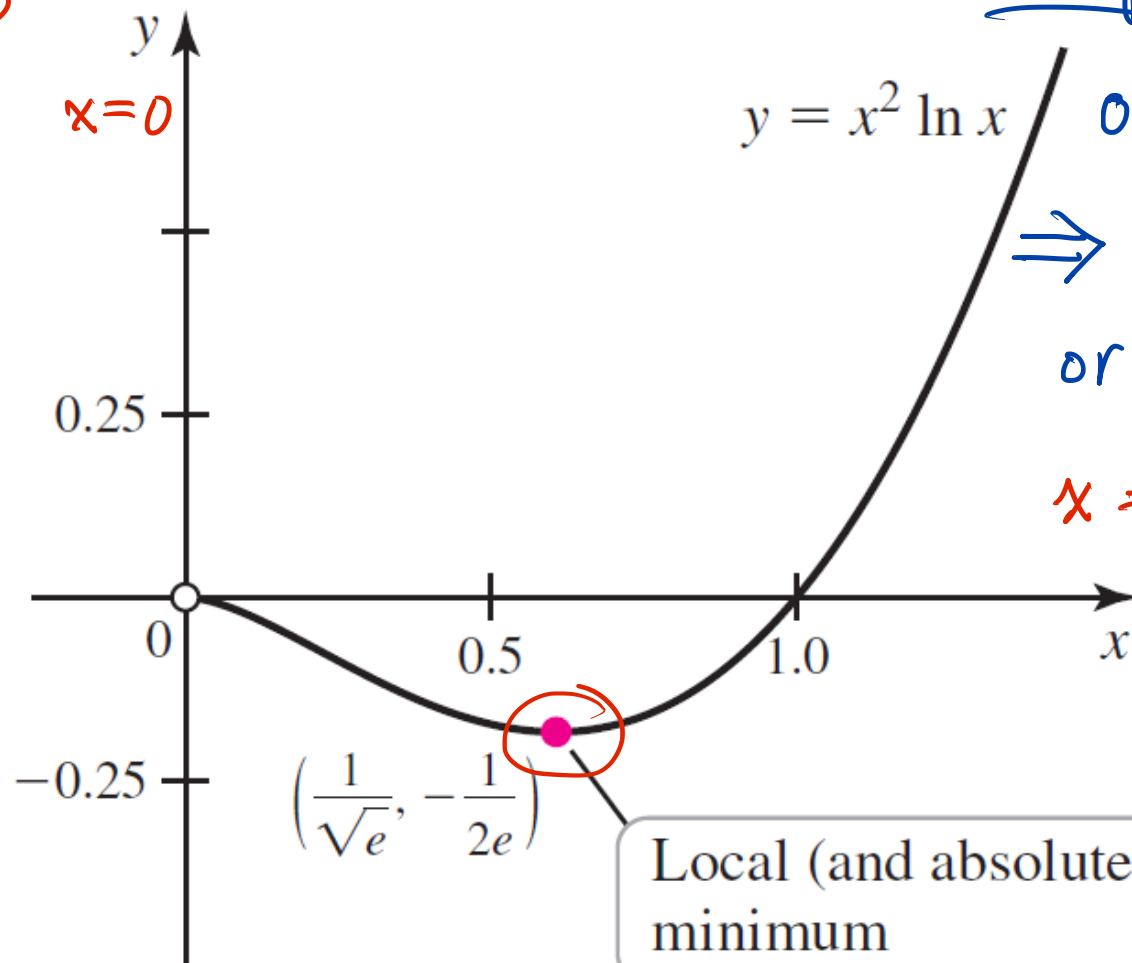
$$\Rightarrow \boxed{x = 0} \quad \text{or} \quad 2 \ln x + 1 = 0$$

$$x = e^{\ln x = -\frac{1}{2}} = e^{-\frac{1}{2}}$$

$$\boxed{x = e^{-\frac{1}{2}}}$$

$$\hat{f}(x) = \begin{cases} x^2 \ln x, & x > 0 \\ 0, & x = 0 \end{cases}$$

$[0, +\infty)$



Procedure Locating Absolute Extreme Values on a Closed Interval

Assume the function f is continuous on the closed interval $[a, b]$.

1. Locate the critical points c in (a, b) , where $f'(c) = 0$ or $f'(c)$ does not exist. These points are candidates for absolute maxima and minima.
2. Evaluate f at the critical points and at the endpoints of $[a, b]$.
3. Choose the largest and smallest values of f from Step 2 for the absolute maximum and minimum values, respectively.

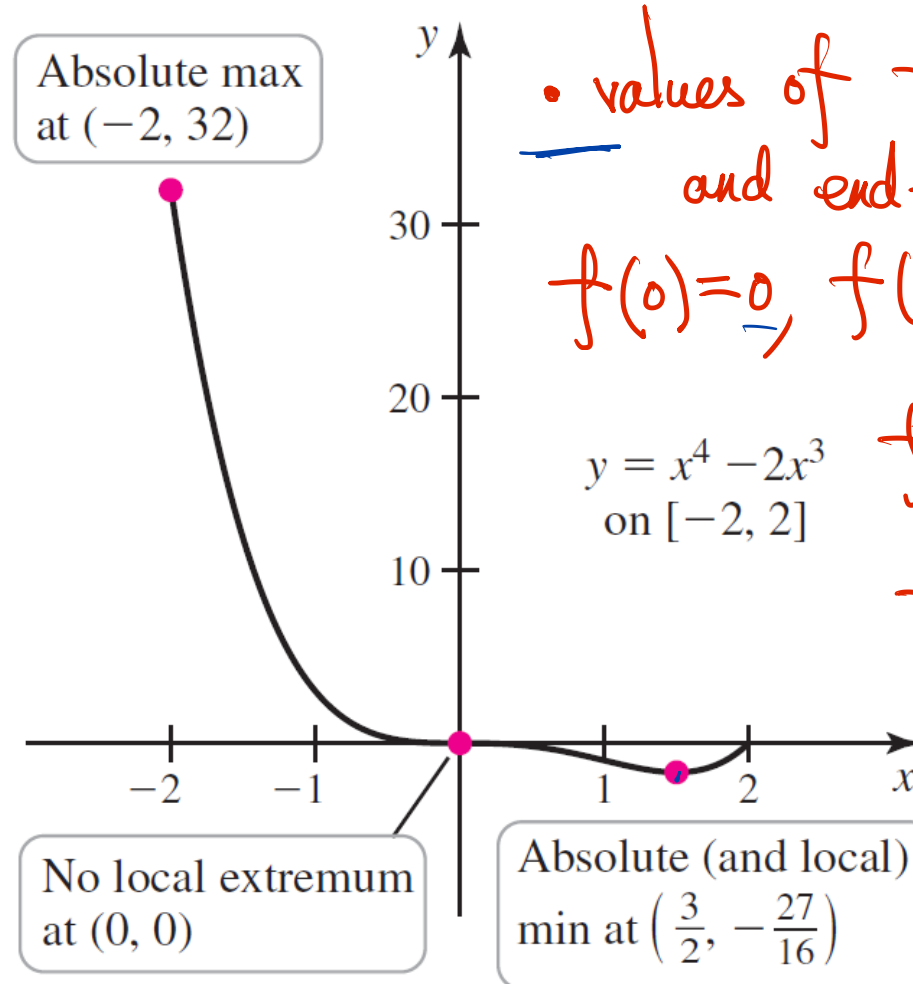
Example 4 Find the abs. extreme values: (a) $f(x) = x^4 - 2x^3$ on $[-2, 2]$

• critical pts

$$0 = f'(x) = 4x^3 - 6x^2 = x^2(4x - 6) = 2x^2(2x - 3)$$

Figure 4.11

$$\Rightarrow \begin{cases} x^2 = 0 \\ \text{or } 2x - 3 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ \text{or } x = \frac{3}{2} \end{cases} \quad \begin{array}{l} \text{critical pts} \\ x = 0 \text{ \& } x = \frac{3}{2} \end{array}$$



• values of $f(x)$ at critical pts and end-pts.

$$f(0) = 0, \quad f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^4 \left(\frac{3}{2} - 2\right) \quad \begin{array}{l} \text{abs.} \\ \text{min} \end{array}$$

$$f(-2) = (-2)^4 - 2 \cdot (-2)^3 = 16 + 16 = 32 \quad \begin{array}{l} \text{abs. max} \end{array}$$

$$f(2) = 2^4 - 2 \cdot 2^3 = 16 - 16 = 0$$

(b) $f(x) = x^{2/3}(2-x)$ on $[-1, 2]$.

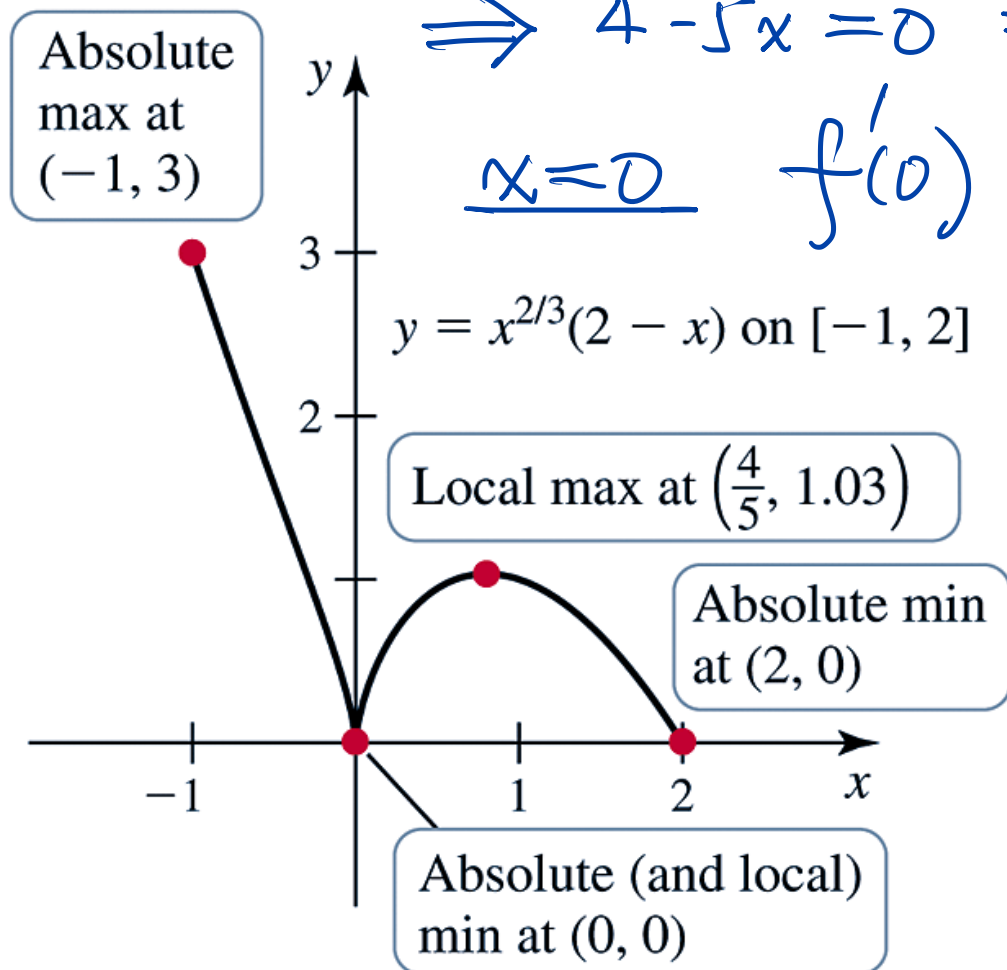
$$f'(x) = \left[2x^{2/3} - x^{5/3} \right]' = 2 \cdot \frac{2}{3} x^{-1/3} - \frac{5}{3} x^{2/3}$$

$$= \frac{1}{3} x^{-1/3} [4 - 5x] = 0$$

Figure 4.12

$$\Rightarrow 4 - 5x = 0 \Rightarrow x = \frac{4}{5}$$

$$\underline{x=0} \quad f'(0) \text{ DNE}$$



$$f\left(\frac{4}{5}\right) = \underline{1.03}$$

$$f(0) = 0$$

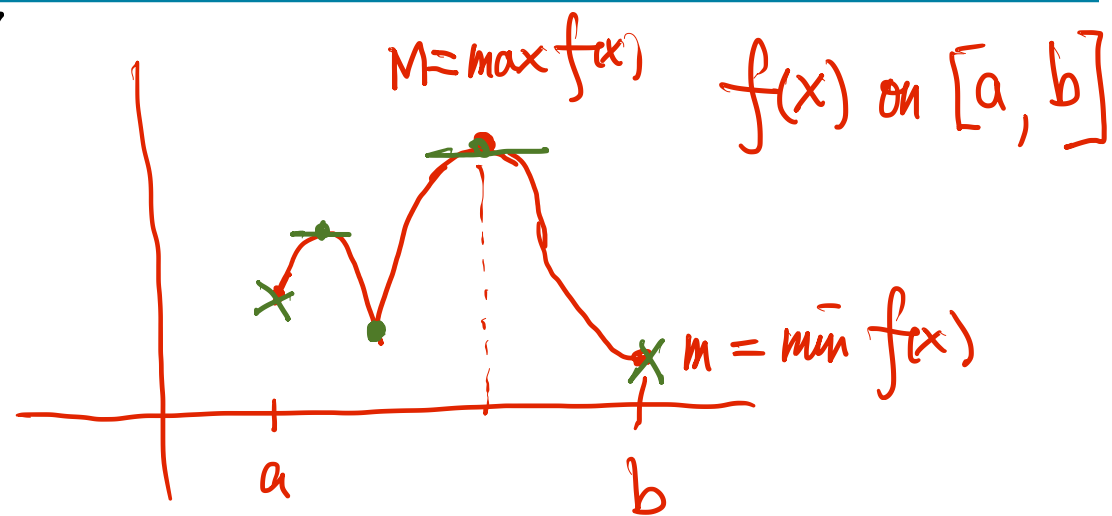
$$f(-1) = 3$$

$$f(2) = 0$$

Section 4.2 Mean Value Theorem

HW# 8, 11, 15, 17, 18, 19, 3, 21, 29, 43, 47

local max./min.
critical pts $f'(c) = 0$
or $f'(c)$ DNE



Rolle's Theorem

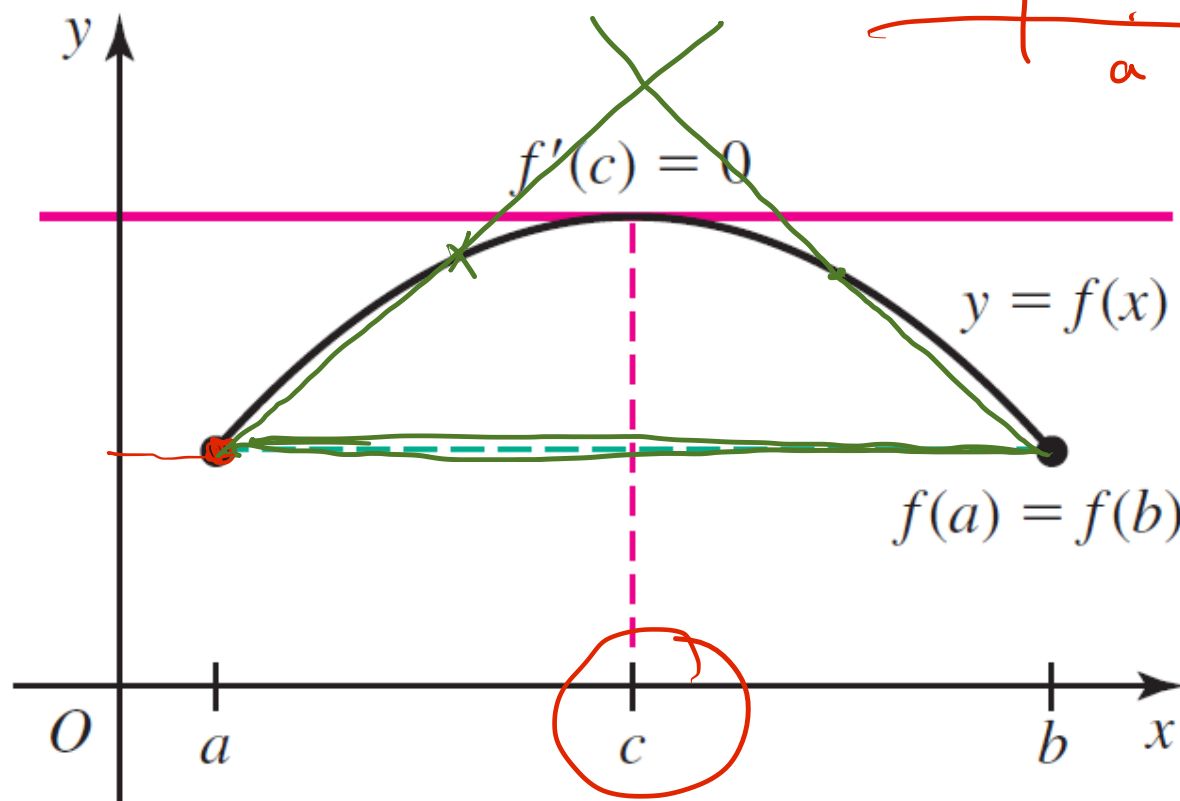
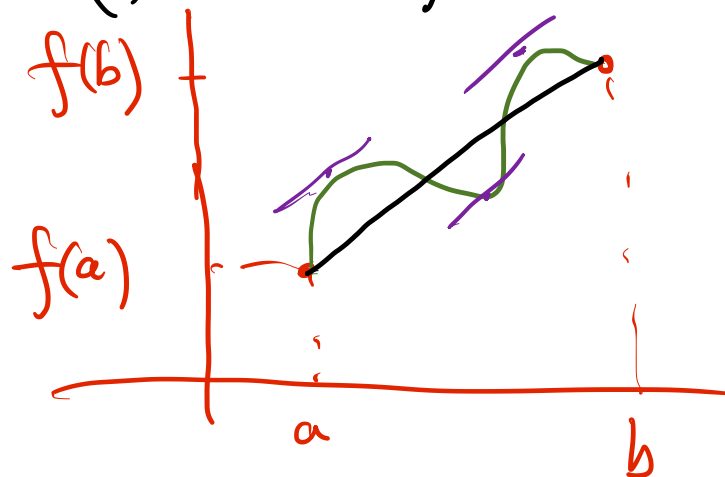
Assumptions

- (a) $f \in C[a, b]$
- (b) $f \in C'(a, b)$
- (c) $f(a) = f(b)$

$$\Rightarrow \exists c \in (a, b) \text{ s.t. } f'(c) = 0.$$

Figure 4.13

$$0 = f'(c) = \frac{f(b) - f(a)}{b - a}$$



Example 1 Verifying Rolle's Thrm for $f(x) = x^3 - 7x^2 + 10x$ on $[0, 5]$.

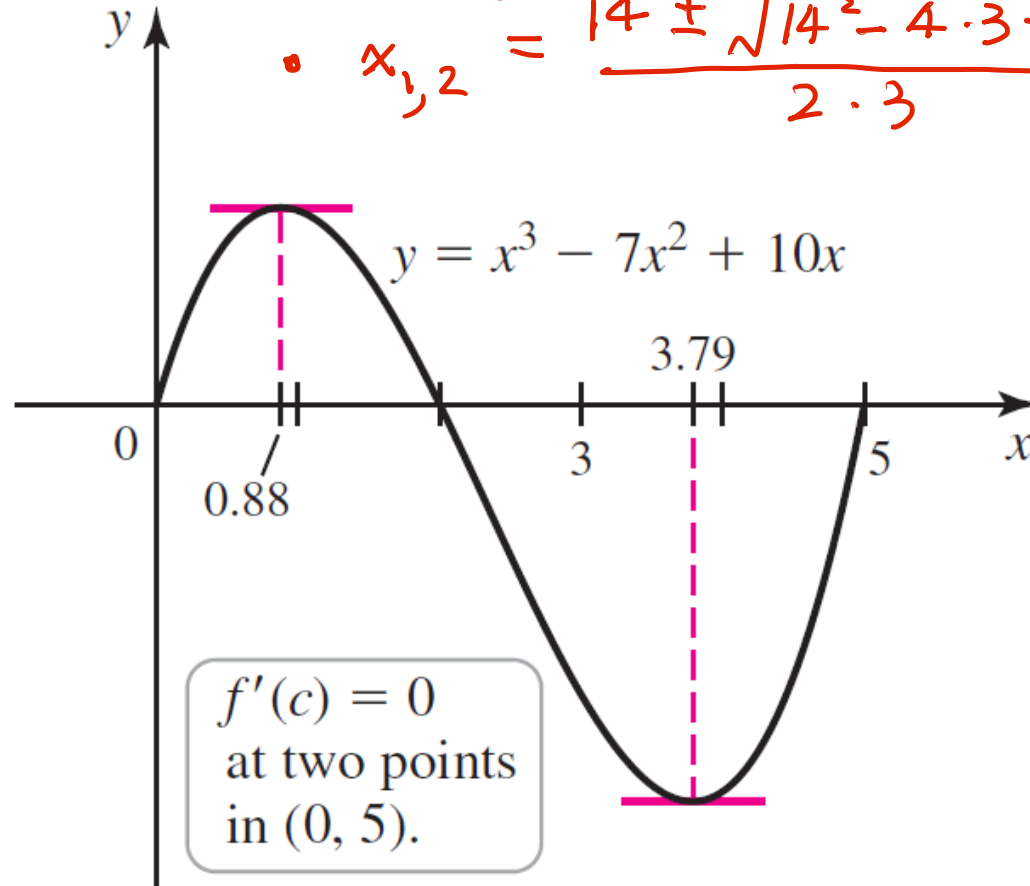
$$f(0) = 0 \text{ \& } f(5) = 0 \xRightarrow{2} \exists c \text{ s.t. } f'(c) = 0$$

Figure 4.15

- $f'(x) = 3x^2 - 14x + 10$

- $0 = f'(x) = 3x^2 - 14x + 10$

- $x_{1,2} = \frac{14 \pm \sqrt{14^2 - 4 \cdot 3 \cdot 10}}{2 \cdot 3} \approx 0.88, 3.79$
 $\in [0, 5]$



Mean Value Theorem

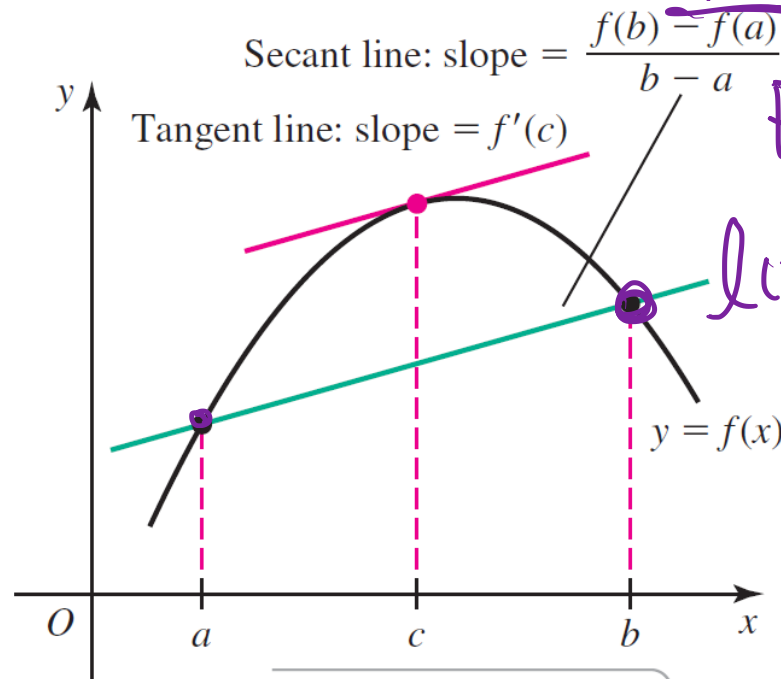
Assumptions

$$\begin{cases} (a) f \in C[a, b] \\ (b) f \in C'(a, b) \end{cases} \Rightarrow \exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \exists c \in (a, b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

Figure 4.16

These lines are parallel and their slopes are equal, that is...



$$\dots \frac{f(b) - f(a)}{b - a} = f'(c)$$

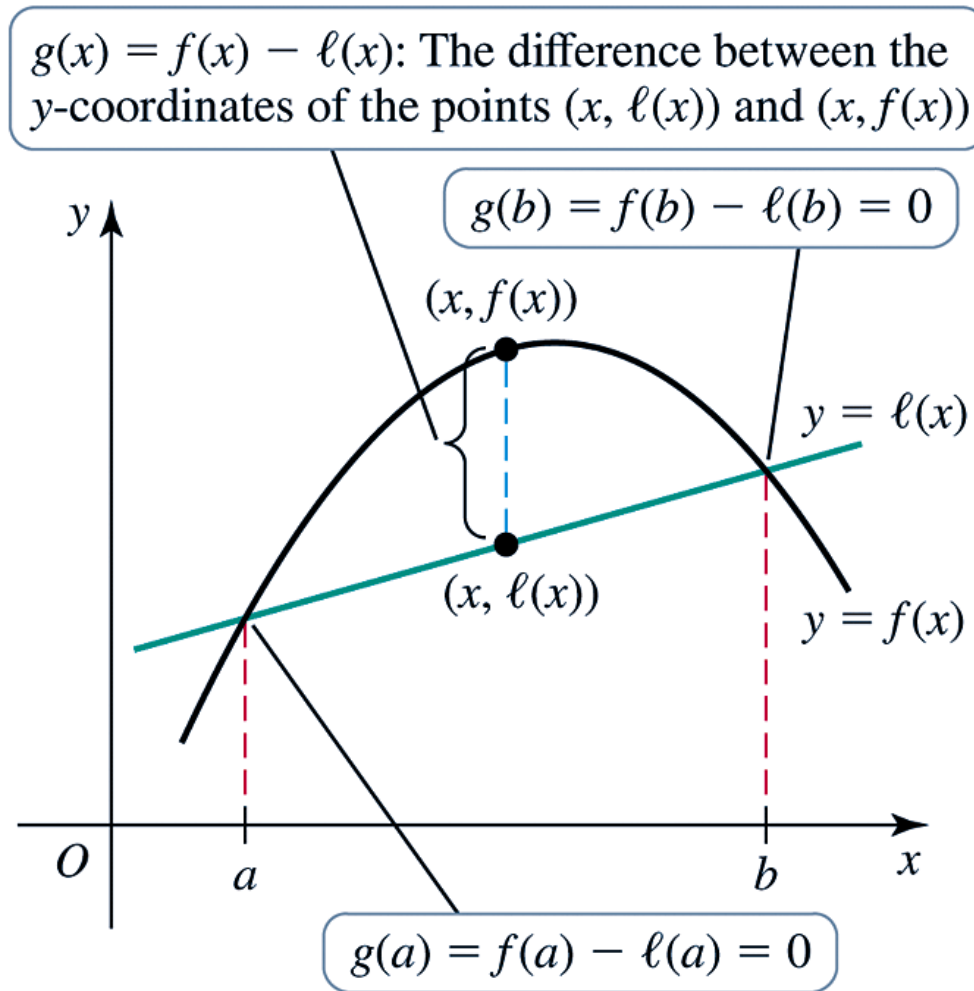
$$f(a) \neq f(b)$$

$$F(x) = f(x) - \underline{\underline{l(x)}}$$

$$F(a) = f(a) - l(a) = 0$$

$$l(x) \quad F(b) = f(b) - l(b) = 0$$

Figure 4.17



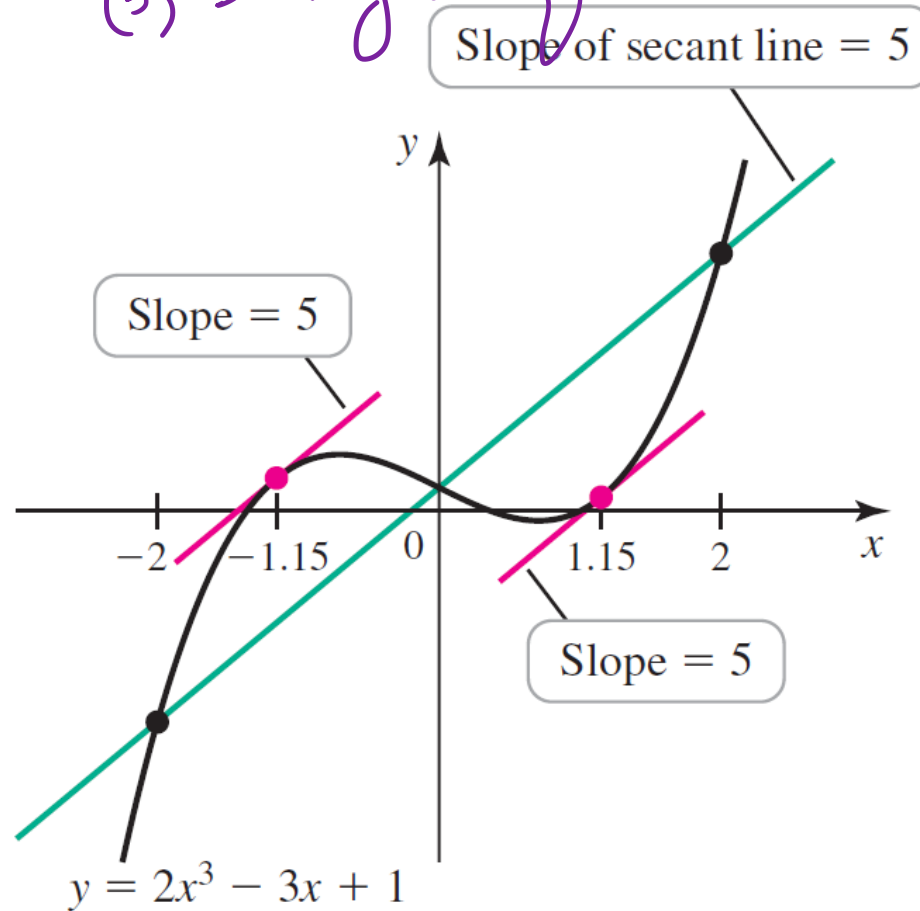
Example 3 Verifying the MVT for $f(x) = 2x^3 - 3x + 1$ on $[-2, 2]$.

(1) $f'(x) = 6x^2 - 3$

Figure 4.19

(2) $f'(x) = 6x^2 - 3 = \frac{f(2) - f(-2)}{2 - (-2)} = 5$

(3) solving the eq. $6x^2 - 3 = 5 \Rightarrow x^2 = \frac{8}{6} = \frac{4}{3}$
 $x = \pm \sqrt{\frac{4}{3}}$



Theorem 4.5 Zero Derivative Implies Constant Function

If f is differentiable and $f'(x) = 0$ at all points of an open interval I , then f is a constant function on I . A. $f(x) \neq C$

$$\underbrace{f'(x) = 0}_{\text{circled}} \quad \forall x \in I \implies f(x) = C \implies \exists x_1 \neq x_2 \in I$$

$$\underbrace{f'(c)}_{\text{underlined}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

$\Leftarrow$
✓

Example show that $\sin^{-1}\left(\frac{2}{5}\right) + \cos^{-1}\left(\frac{2}{5}\right) = \frac{\pi}{2}$.

Theorem 4.6 Functions with Equal Derivatives Differ by a Constant

If two functions have the property that $f'(x) = g'(x)$, for all x of an open interval I , then $f(x) - g(x) = C$ on I , where C is a constant; that is, f and g differ by a constant.

$$(f - g)' = f' - g' = 0 \implies f - g = C$$

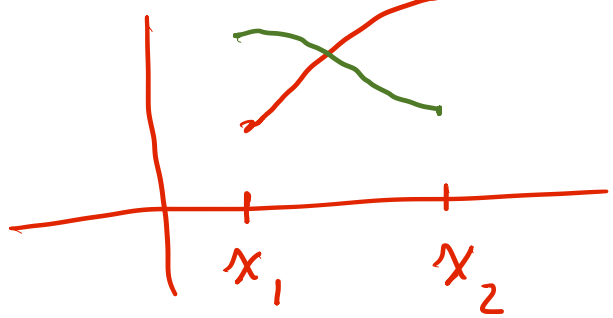
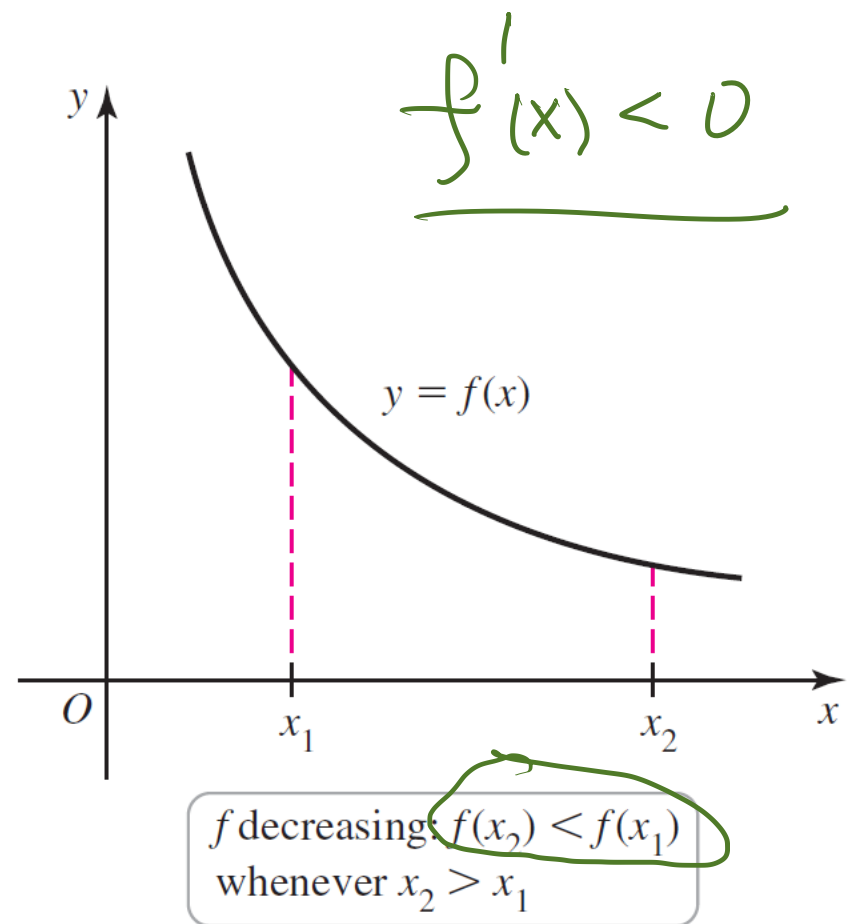
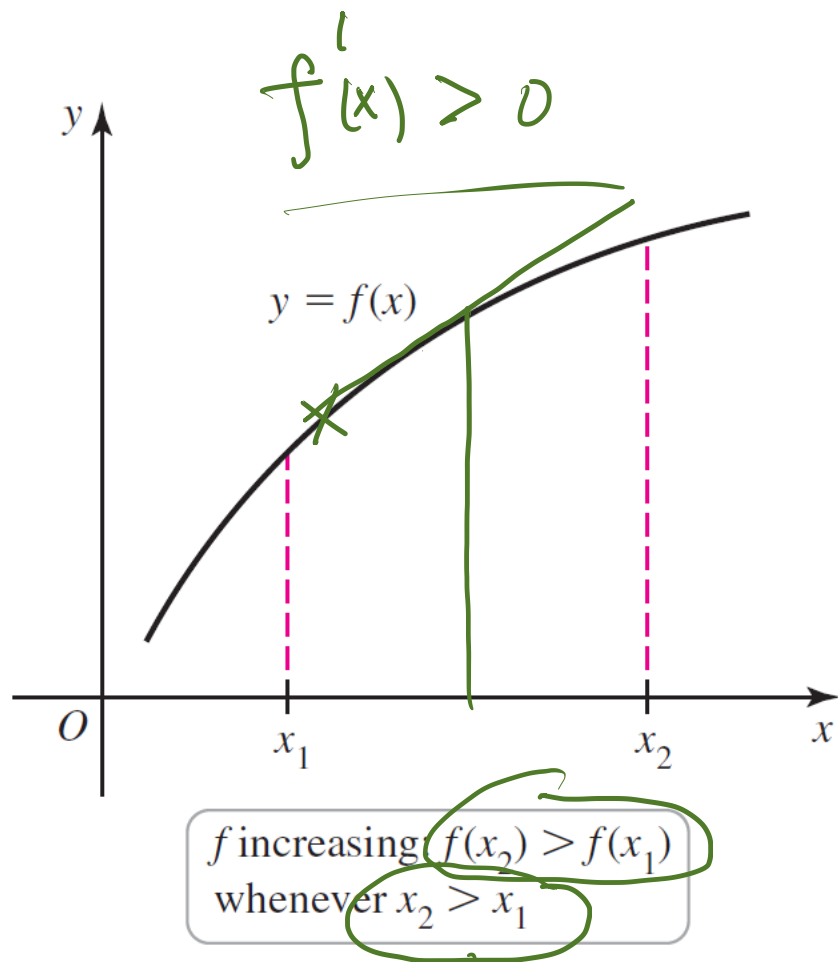
Section 4.3 What Derivatives Tell Us

Part I HW# 1, 9, 25, 29, 31, 34, 39, 46, 48, 77, 90

Part II HW# 15, 16, 17, 59, 63, 65, 71, 76, 81, 96, 101, 113

- increasing and decreasing intervals of functions
- local maxima and minima.
- concavity and inflection points

Figure 4.20 (a & b)



Theorem 4.7 Test for Intervals of Increase and Decrease

Suppose f is continuous on an interval I and differentiable at all interior points of I . If $f'(x) > 0$ at all interior points of I , then f is increasing on I . If $f'(x) < 0$ at all interior points of I , then f is decreasing on I .

Proof " $f'(x) > 0 \quad \forall x \in I \implies f$ is increasing on I "

Example 1 Sketching a graph of a function that is continuous on $(-\infty, +\infty)$ and satisfying the following conditions.

Figure 4.22

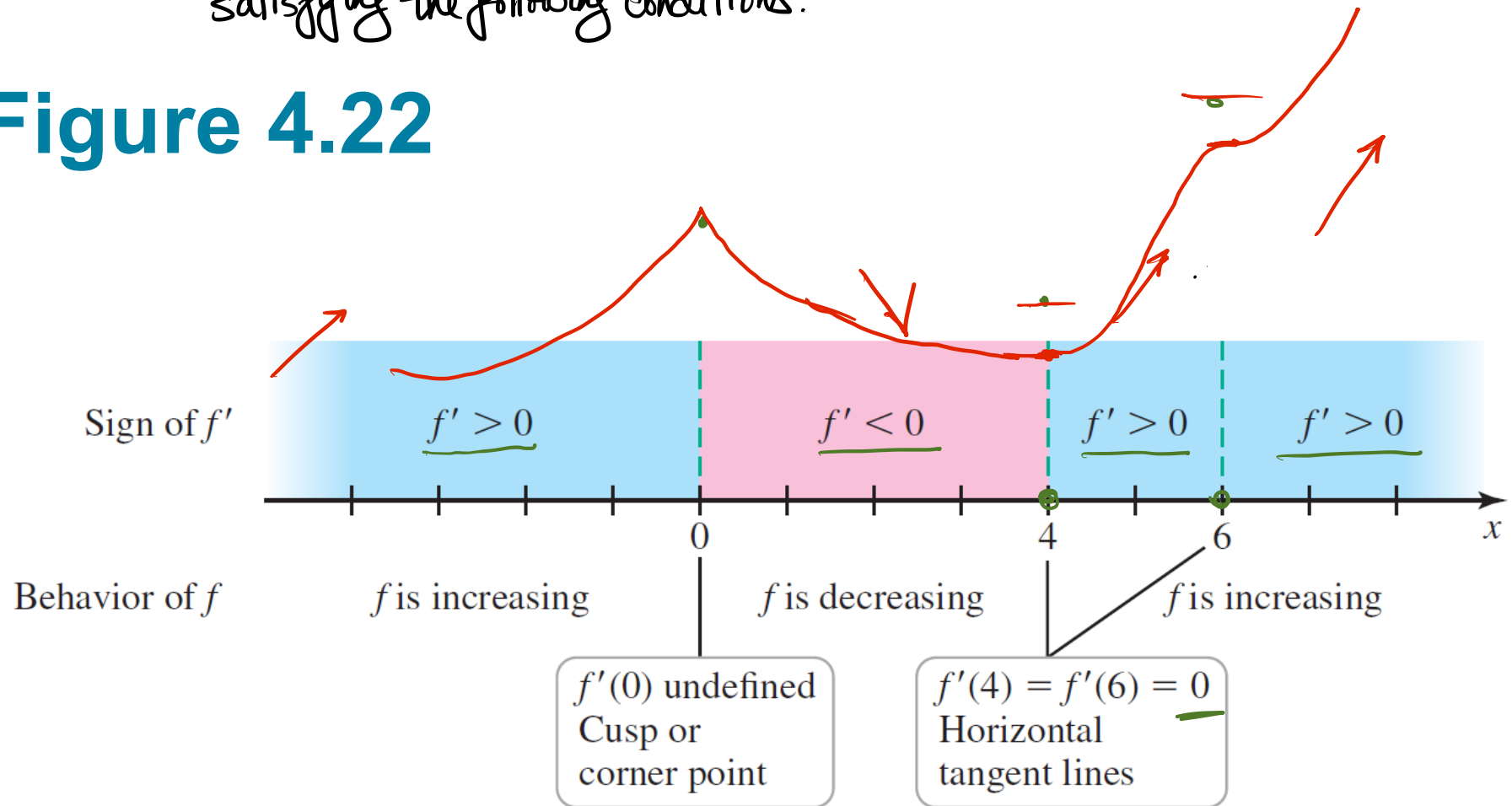
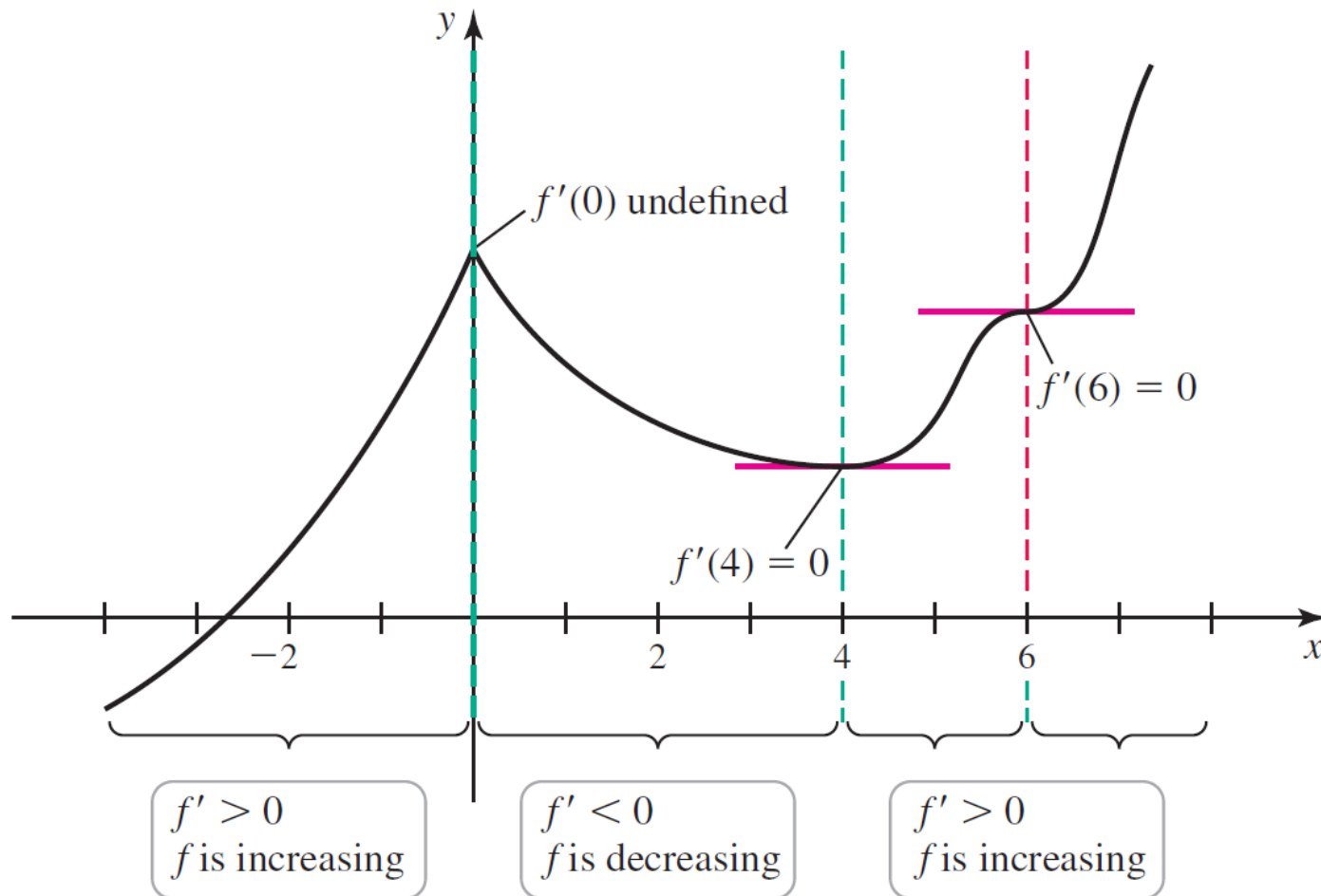


Figure 4.23



Example 2 Find intervals on which functions are increasing or decreasing.

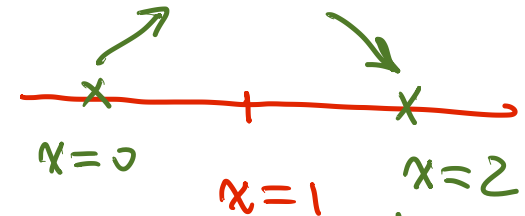
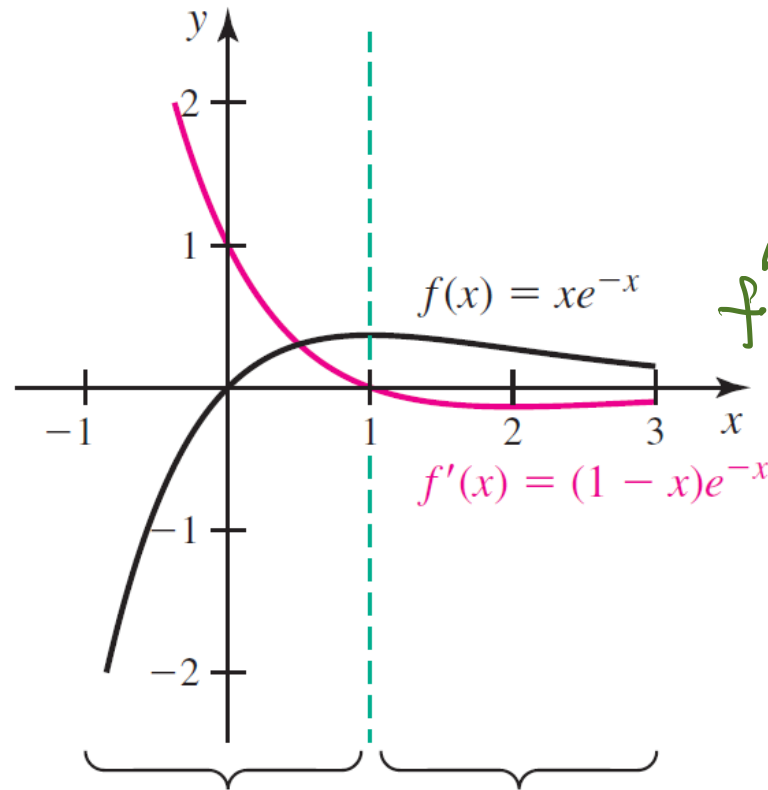
(a) $f(x) = xe^{-x}$

Figure 4.24

$$(1) f'(x) = e^{-x} + x(e^{-x})' = e^{-x} + x e^{-x} \cdot (-x)'$$

$$f'(x) = (1-x)e^{-x}$$

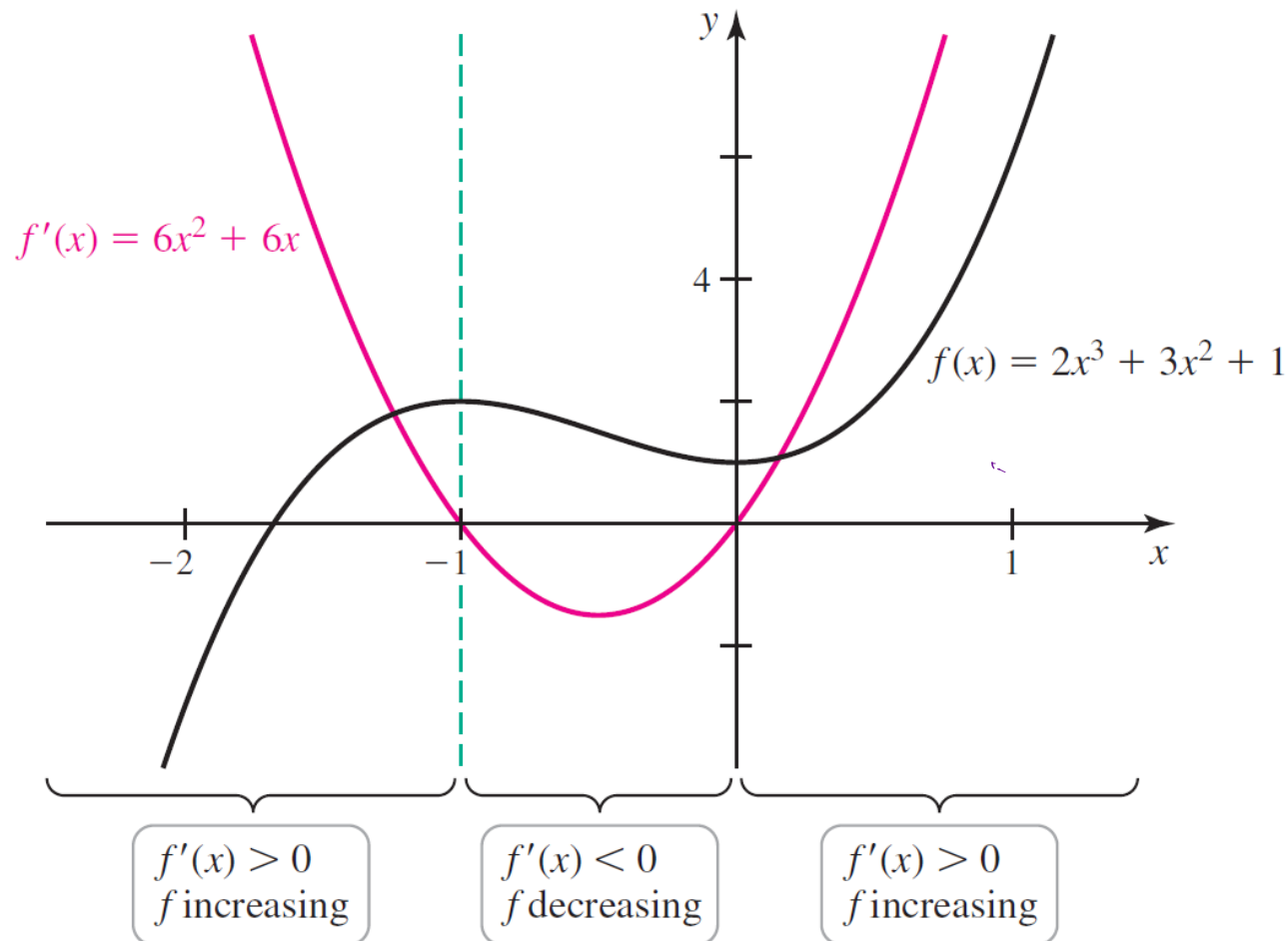
$$(2) 0 = f'(x) = (1-x)e^{-x} \Rightarrow 1-x=0 \Rightarrow \underline{x=1}$$



$$f'(0) = e^0 = 1 > 0 \quad f'(2) = (1-2)e^{-2} = -e^{-2} < 0$$

(b) $f(x) = 2x^3 + 3x^2 + 1$.

Figure 4.25



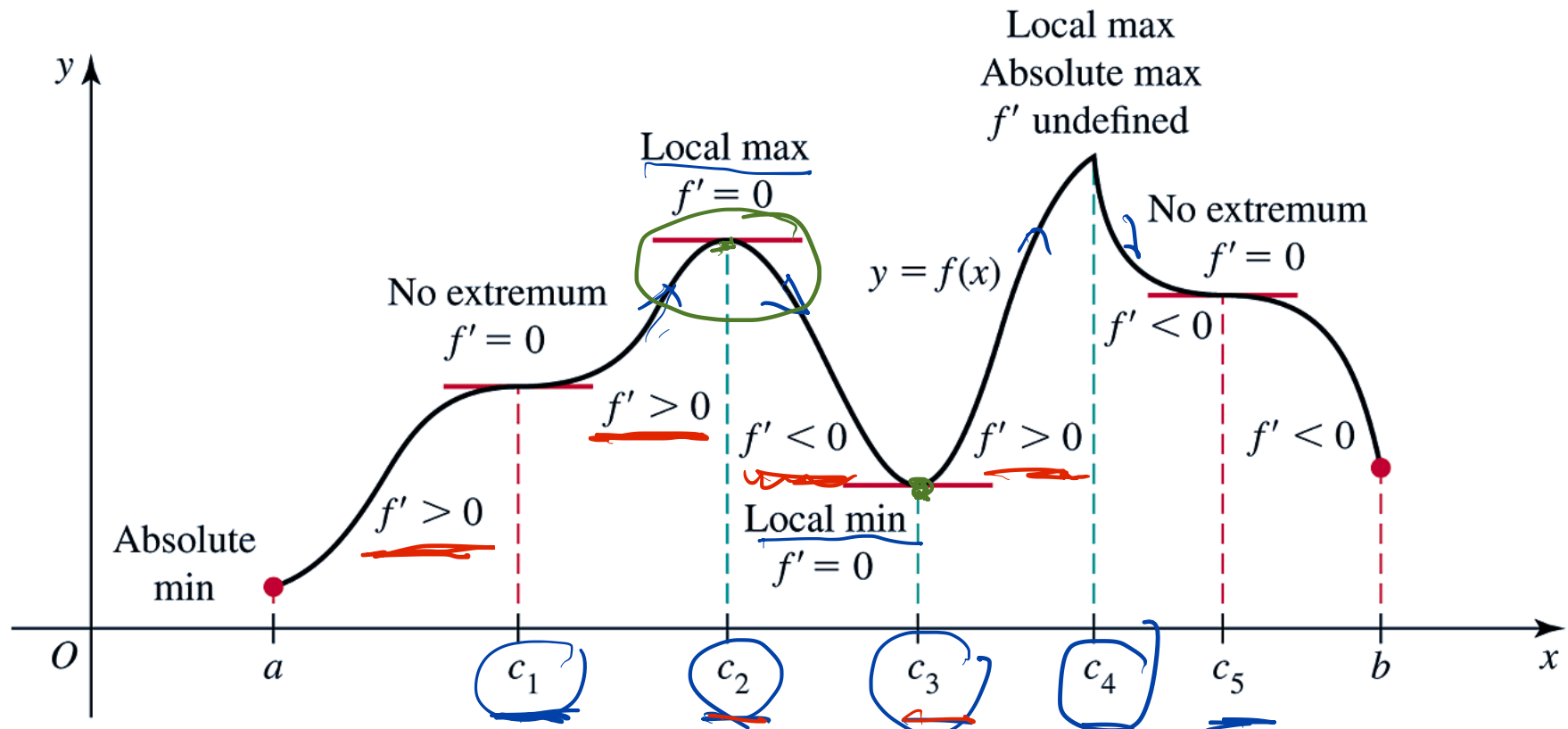
Identifying Local Maxima and Minima

1st-der. and 2nd-der. tests

Figure 4.27

based on critical pts

$$f'(c) = 0 \text{ or } f'(c) \text{ DNE}$$



Theorem 4.8 First Derivative Test

Assume f is continuous on an interval that contains a critical point c , and assume f is differentiable on an interval containing c , except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c , then f has a **local maximum** at c .
- If f' changes sign from negative to positive as x increases through c , then f has a **local minimum** at c .
- If f' is positive on both sides near c or negative on both sides near c , then f has no local extreme value at c .

Example 3 $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$. Local extrema?

$$f'(x) = 12x^3 - 12x^2 - 12x + 12 = 12 \left[\underbrace{x^3 - x^2}_{x^2(x-1)} - \underbrace{x + 1}_{-(x-1)} \right] = 0$$

$$x^2(x-1) - (x-1) = (x-1)(x^2 - 1) = (x-1)^2(x+1)$$

cp $x = 1, -1$



Sign of
 $f'(x) = 12(x+1)(x-1)^2$

Behavior of f

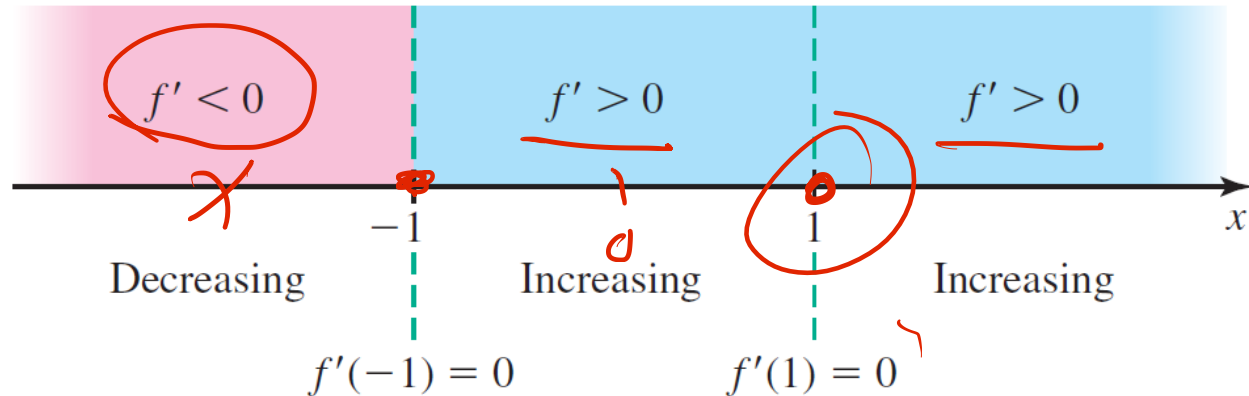
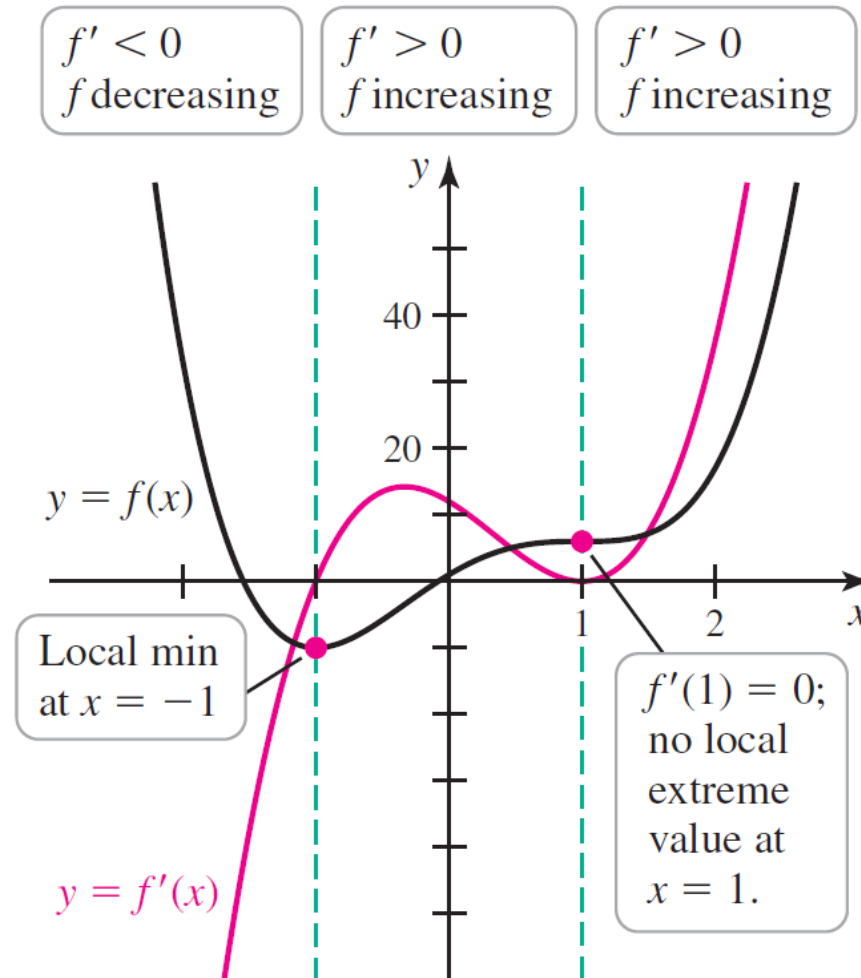


Figure 4.29



Example 4 $g(x) = x^{\frac{2}{3}}(2-x)$.

Figure 4.30

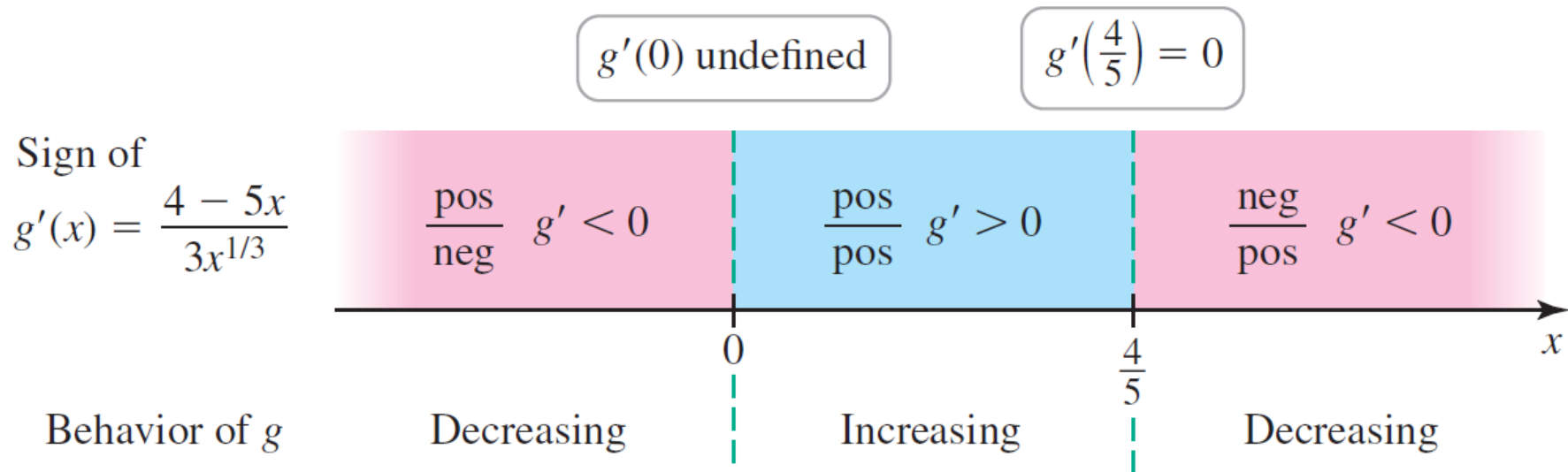
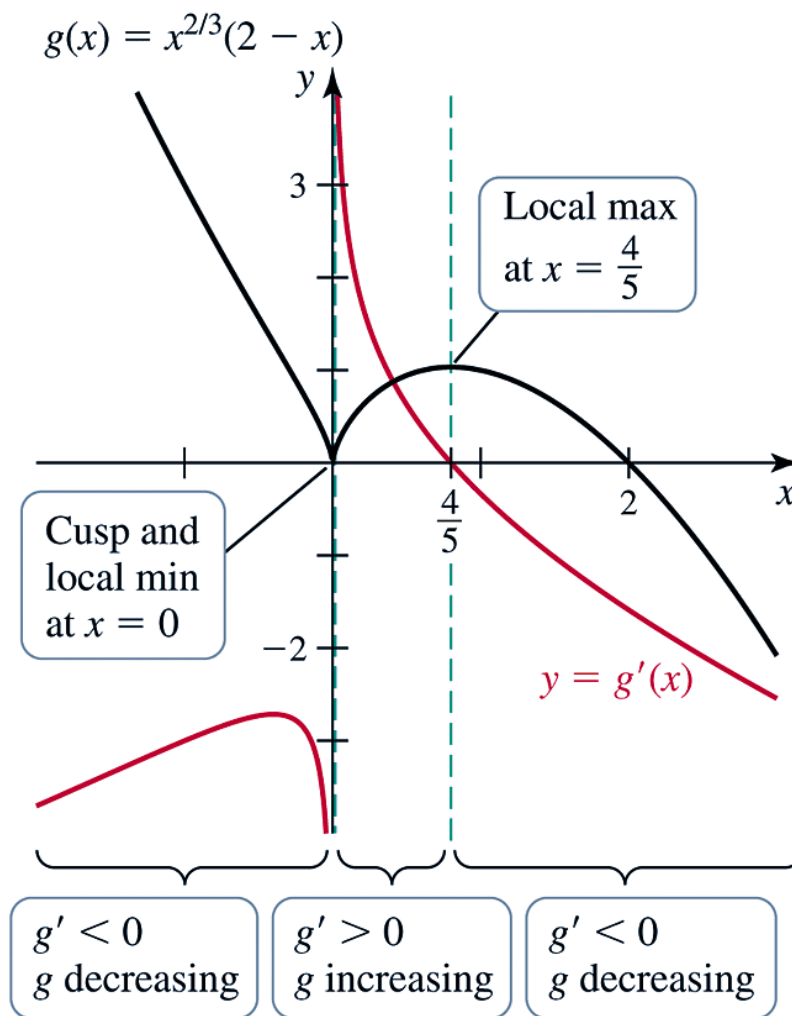
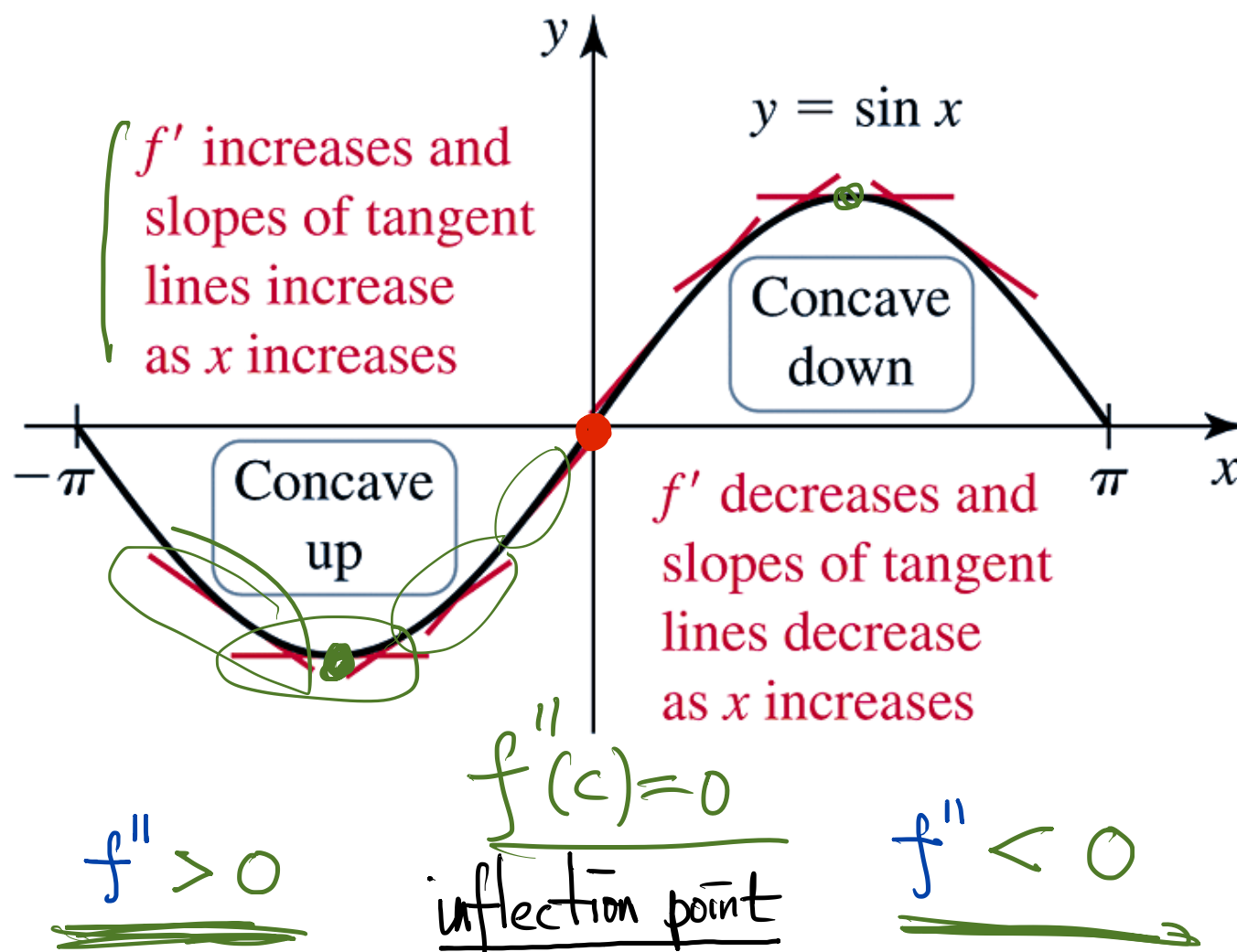


Figure 4.31



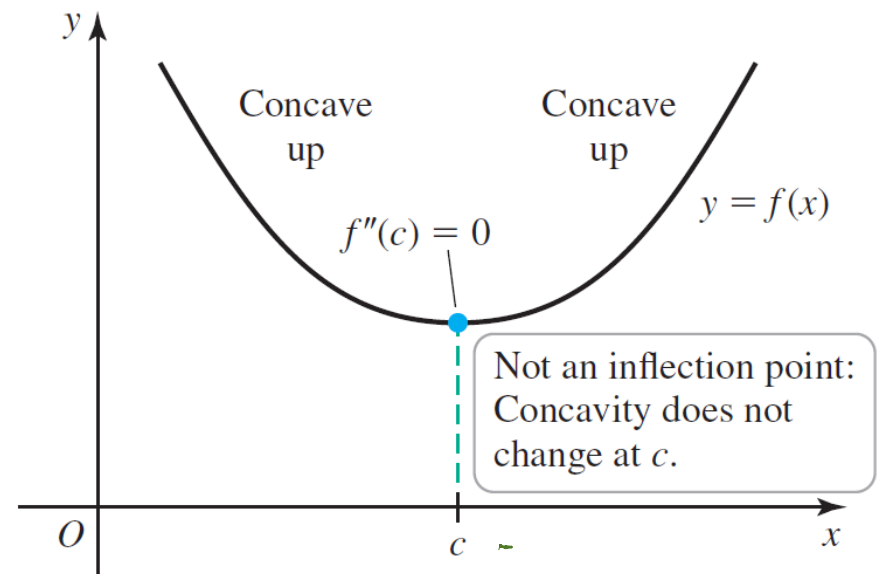
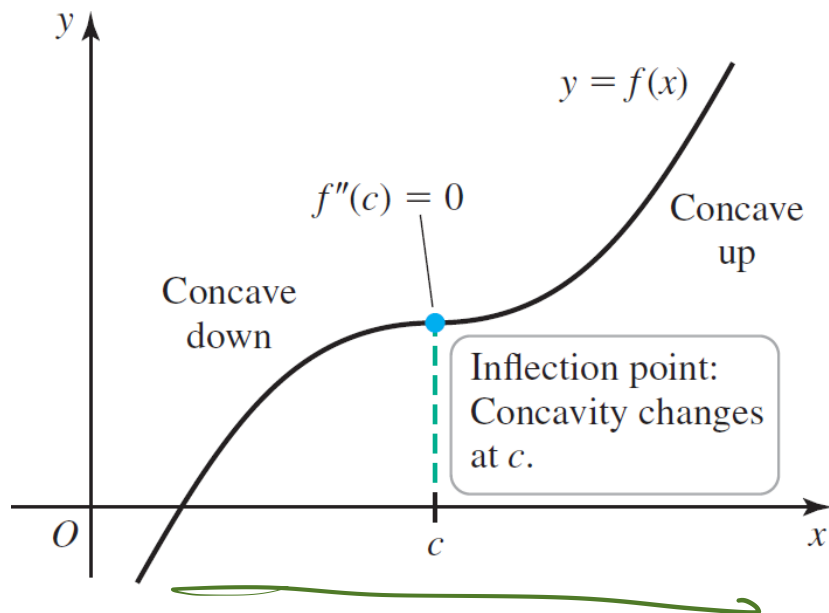
- Concavity and Inflection Points

Figure 4.34



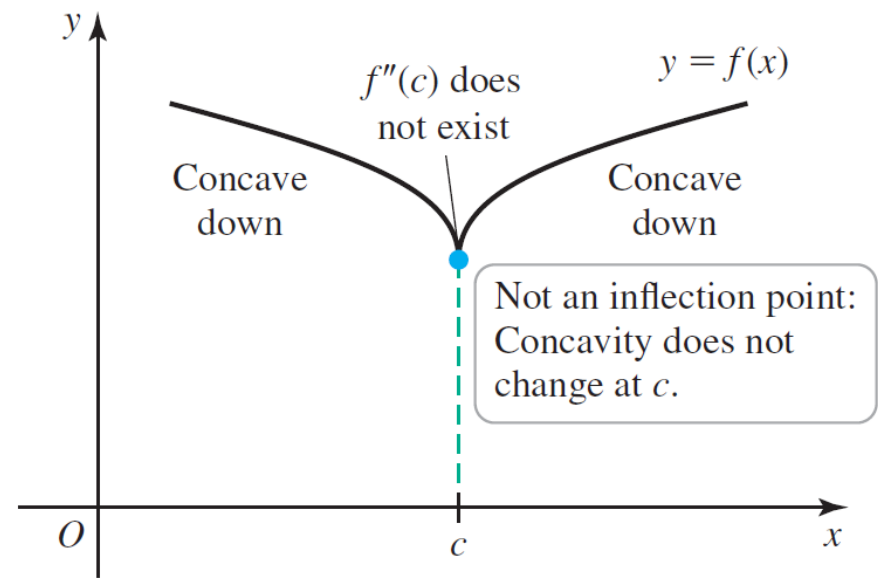
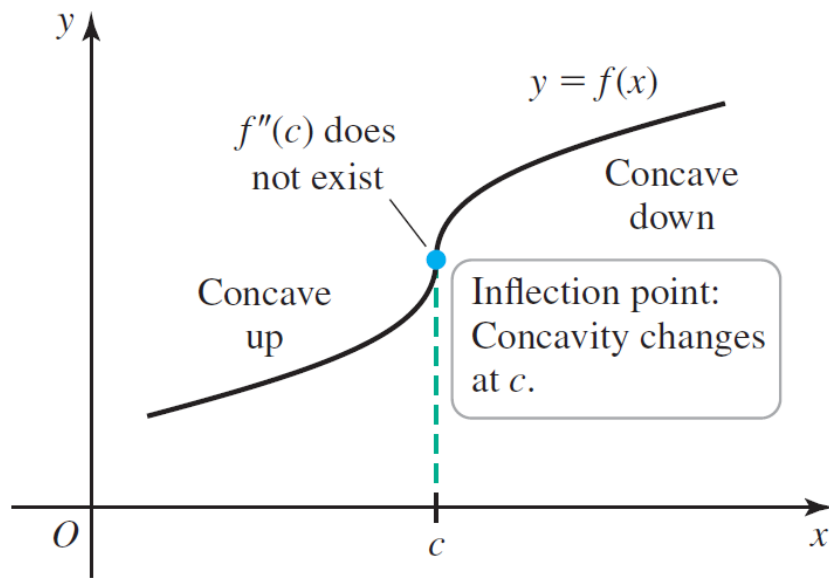
$f''(c) = 0 \implies f$ has an inflection pt at c

Figure 4.35 (a & b)



$f''(c)$ DNE $\Rightarrow$ f has an inflection pt at c

Figure 4.35 (c & d)



Example 7 $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$ (detecting concavity)

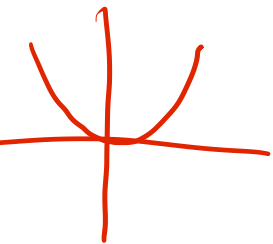
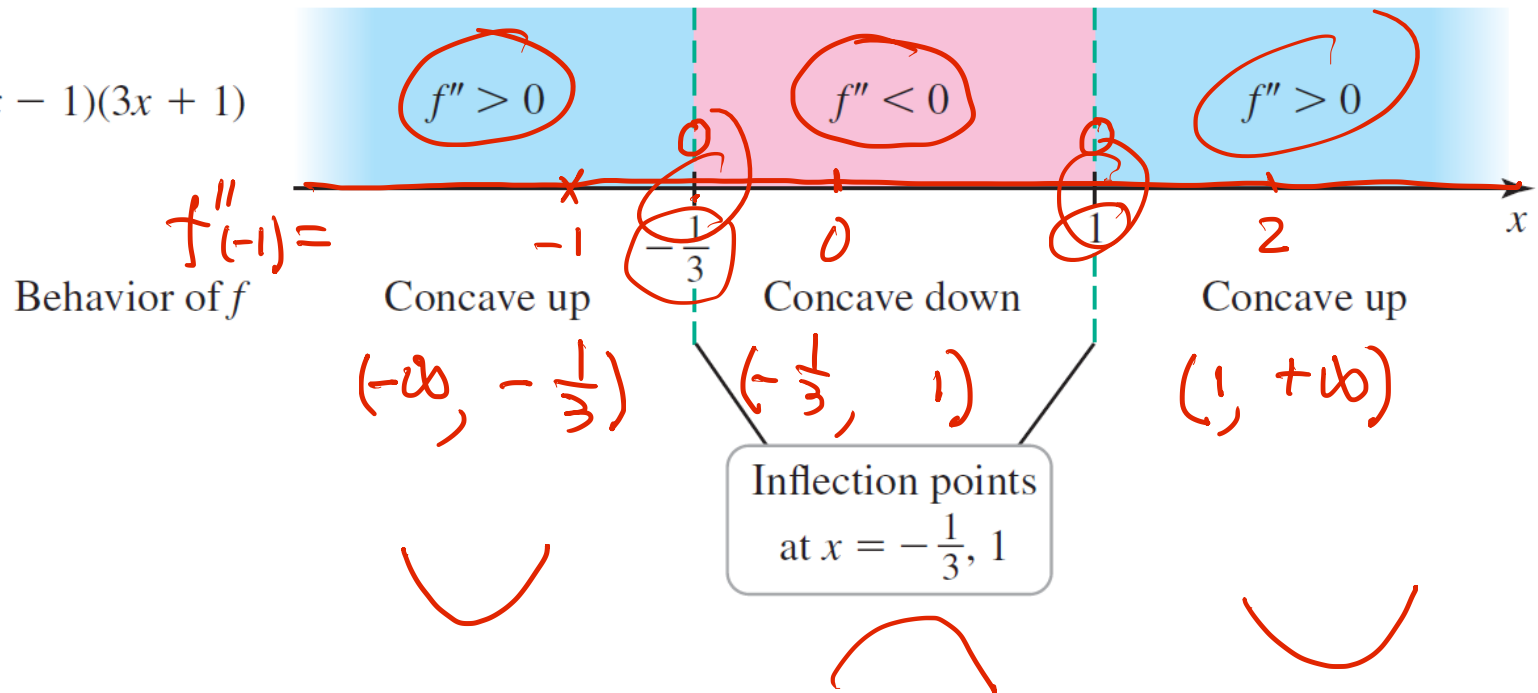
$$f'(x) = 12x^3 - 12x^2 - 12x + 12 = 12[x^3 - x^2 - x + 1]$$

Figure 4.37 $f''(x) = 12[3x^2 - 2x - 1] = 0$
 $= 12(x-1)(3x+1) \Rightarrow x=1$
 or $x = -\frac{1}{3}$

$$f''(-\frac{1}{3}) = 0$$

$$f''(1) = 0$$

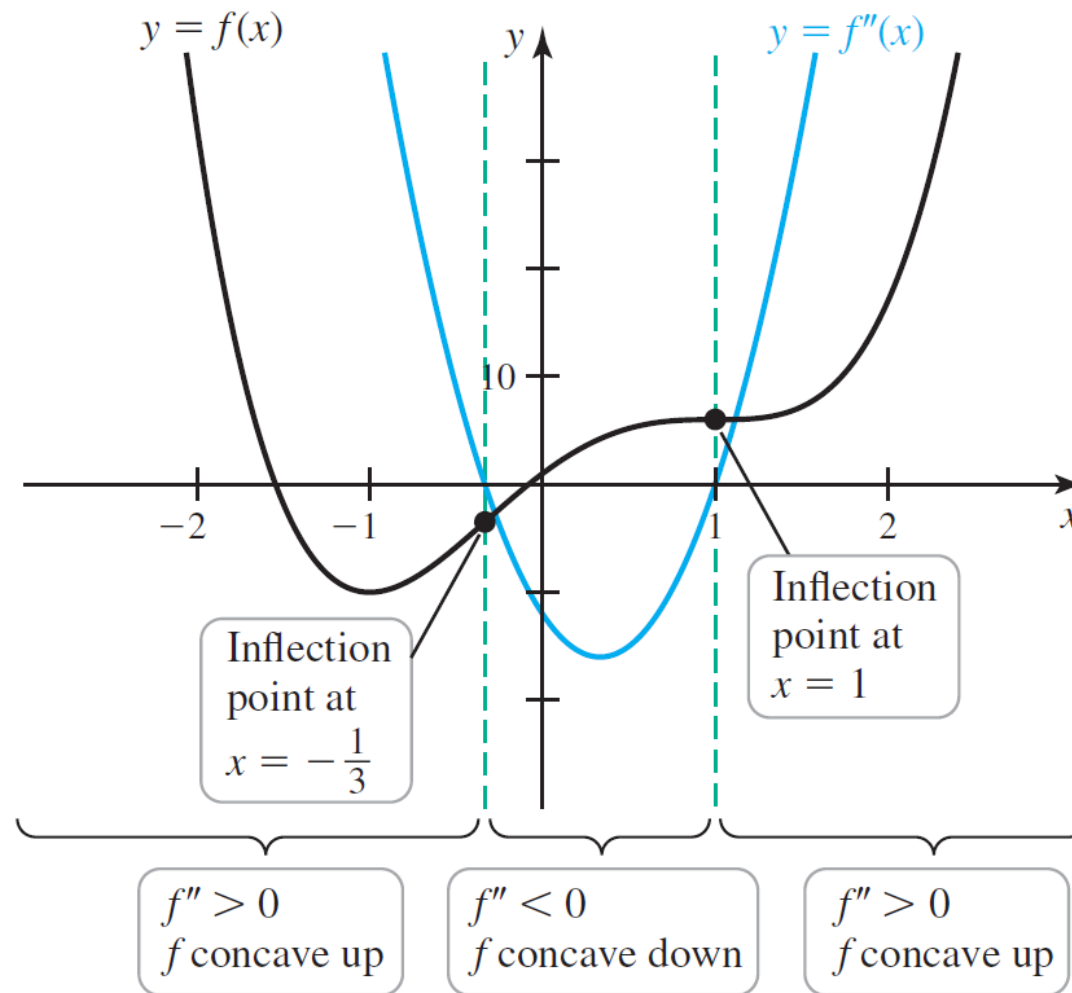
Sign of
 $f''(x) = 12(x-1)(3x+1)$



$$f(x) = x^2$$

$$f'' = 2 > 0$$

Figure 4.38



Theorem 4.11 Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c (**Figure 4.40a**).
- If $f''(c) < 0$, then f has a local maximum at c (**Figure 4.40b**).
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, a local minimum, or neither at c .

Example 8 Use the 2nd-derivative test to locate the local extrema.

$$f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1 \text{ on } [-2, 2]$$

Figure 4.41

$$f'(x) = 12(x+1)(x-1)^2 \text{ and } f''(x) = 12(x-1)(3x+1)$$

$$c = -1, 1$$

$$f''(-1) > 0 \Rightarrow \text{local min}$$

$$f''(1) = 0 \text{ ?}$$

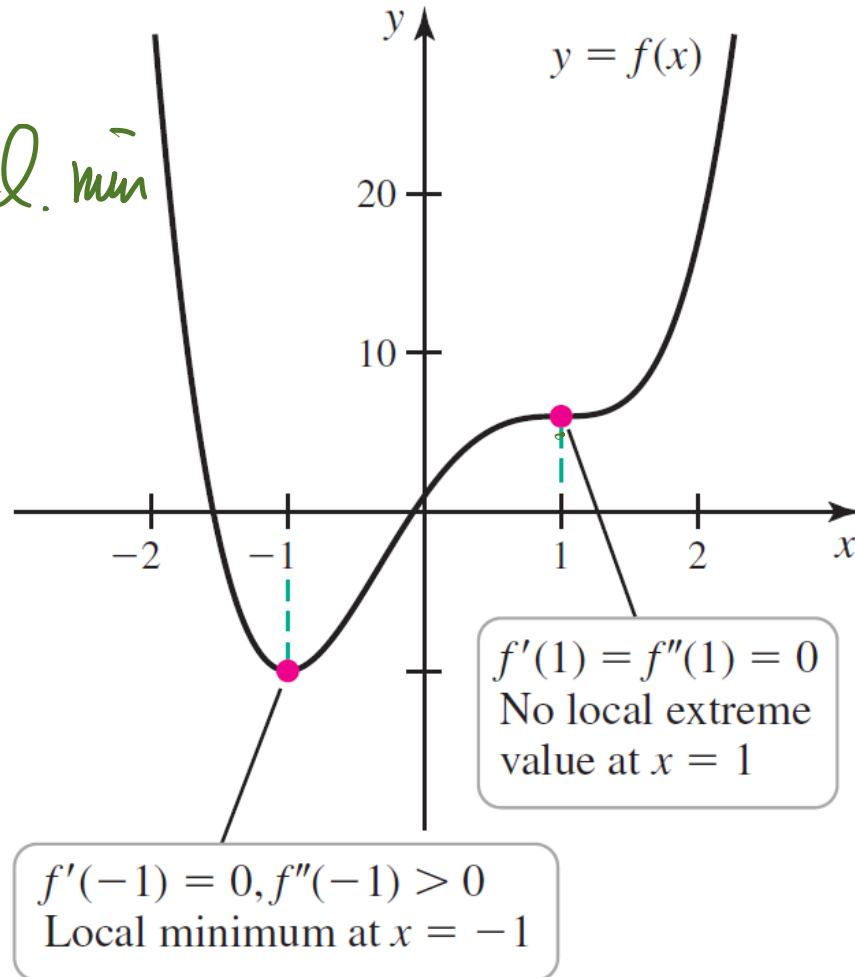
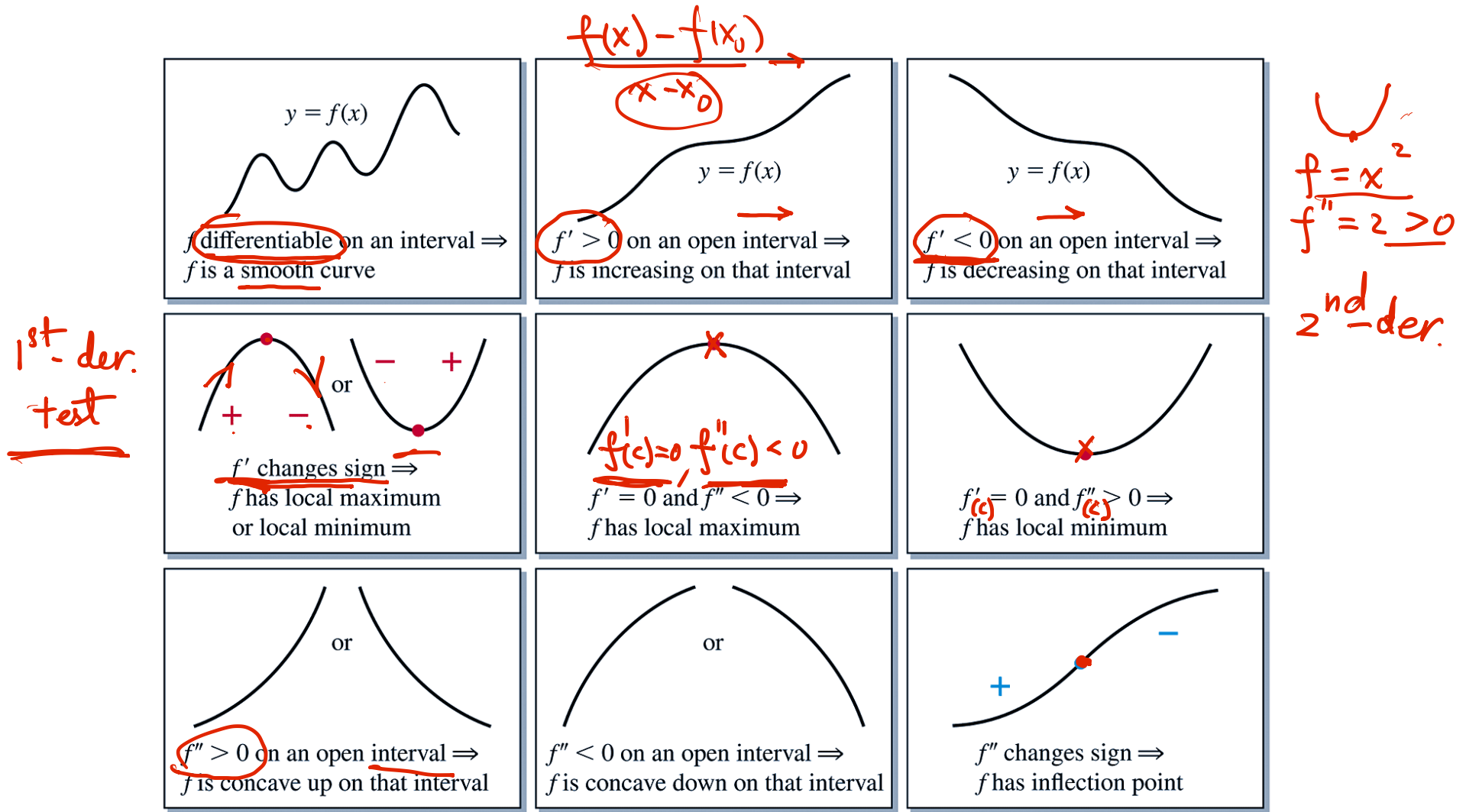


Figure 4.43



Section 4.4 Graphing Functions

HW # 8, 25, 29, 33, 35, 37, 42, 45, 47, 49, 53, 55, 56, 57, 59, 61, 63, 64.

Graphing Guidelines for $y = f$ of x (1 of 2)

1. **Identify the domain or interval of interest.** On what interval(s) should the function be graphed? It may be the domain of the function or some subset of the domain.
- ? 2. **Exploit symmetry.** Take advantage of symmetry. For example, is the function even ($f(-x) = f(x)$), odd ($f(-x) = -f(x)$), or neither?
- 3. **Find the first and second derivatives.** They are needed to determine extreme values, concavity, inflection points, and intervals of increase and decrease. Computing derivatives-particularly second derivatives-may not be practical, so some functions may need to be graphed without complete derivative information.
- 4. **Find critical points and possible inflection points.** Determine points at which $f'(x) = 0$ or f' is undefined. Determine points at which $f''(x) = 0$ or f'' is undefined

Graphing Guidelines for $y = f$ of x (2 of 2)

- **5. Find intervals on which the function is increasing/decreasing and concave up/down.** The first derivative determines the intervals of increase and decrease. The second derivative determines the intervals on which the function is concave up or concave down.
- **6. Identify extreme values and inflection points.** Use either the First or Second Derivative Test to classify the critical points. Both x - and y -coordinates of maxima, minima, and inflection points are needed for graphing.
- **7. Locate all asymptotes and determine end behavior.** Vertical asymptotes often occur at zeros of denominators. Horizontal asymptotes require examining limits as $x \rightarrow \pm\infty$; these limits determine end behavior. Slant asymptotes occur with rational functions in which the degree of the numerator is one more than the degree of the denominator.
- 8. Find the intercepts.** The y -intercept of the graph is found by setting $x = 0$. The x -intercepts are found by solving $f(x) = 0$; they are the real zeros (or roots) of f .
- 9. Choose an appropriate graphing window and plot a graph.** Use the results of the previous steps to graph the function. If you use graphing software, check for consistency with your analytical work. Is your graph **complete**-that is, does it show all the essential details of the function?

Example 1 (a warm-up)

Figure 4.45

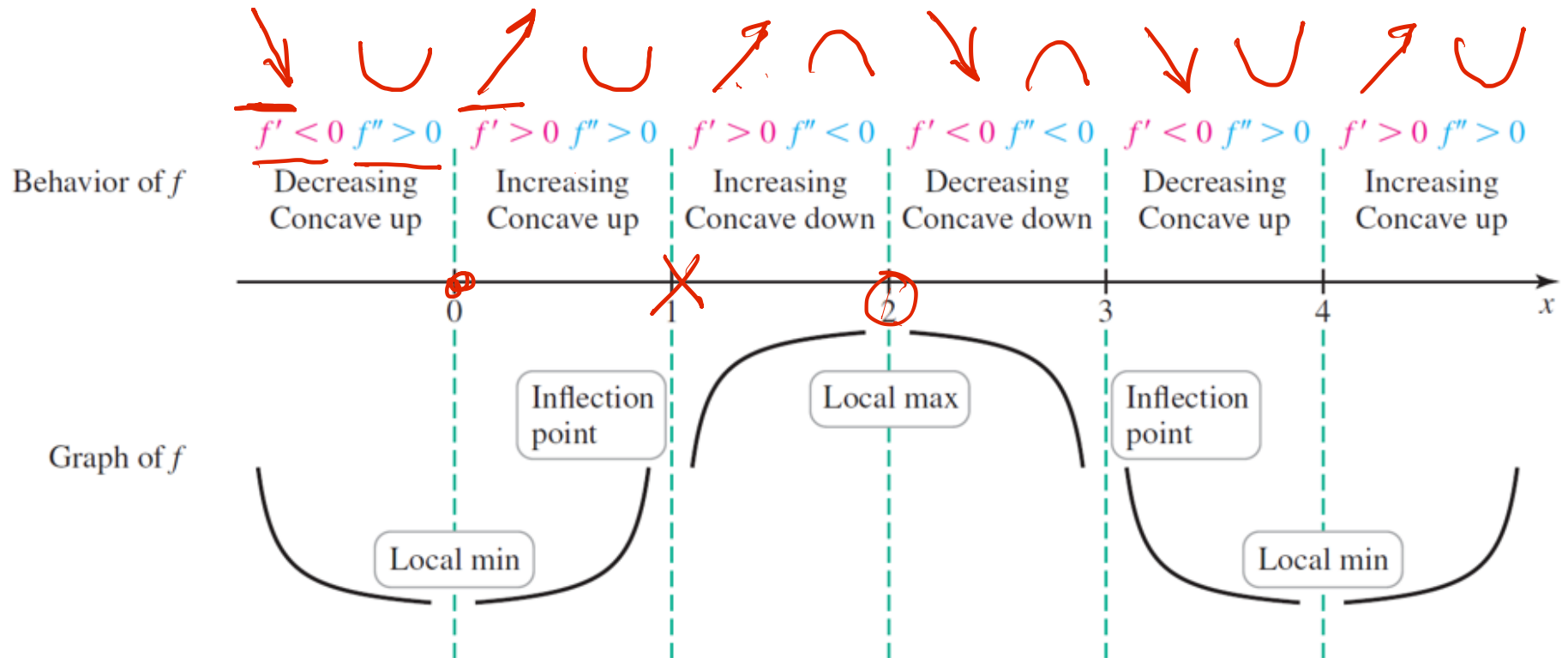
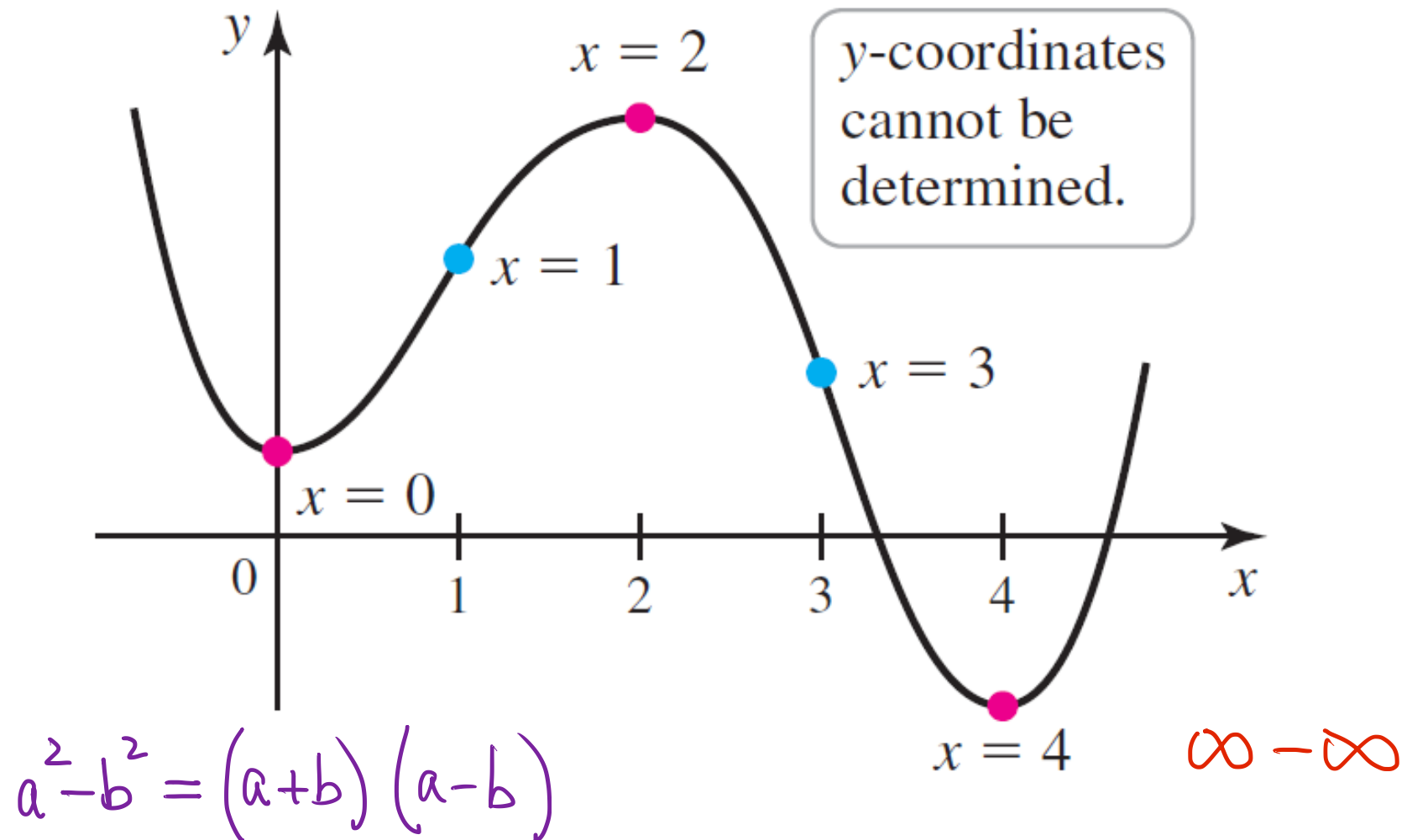


Figure 4.46



Example 2

$$f(x) = \frac{1}{3}x^3 - 400x^0$$

(1) domain $(-\infty, \infty)$

(2) symmetry

$$(3) f'(x) = x^2 - 400$$

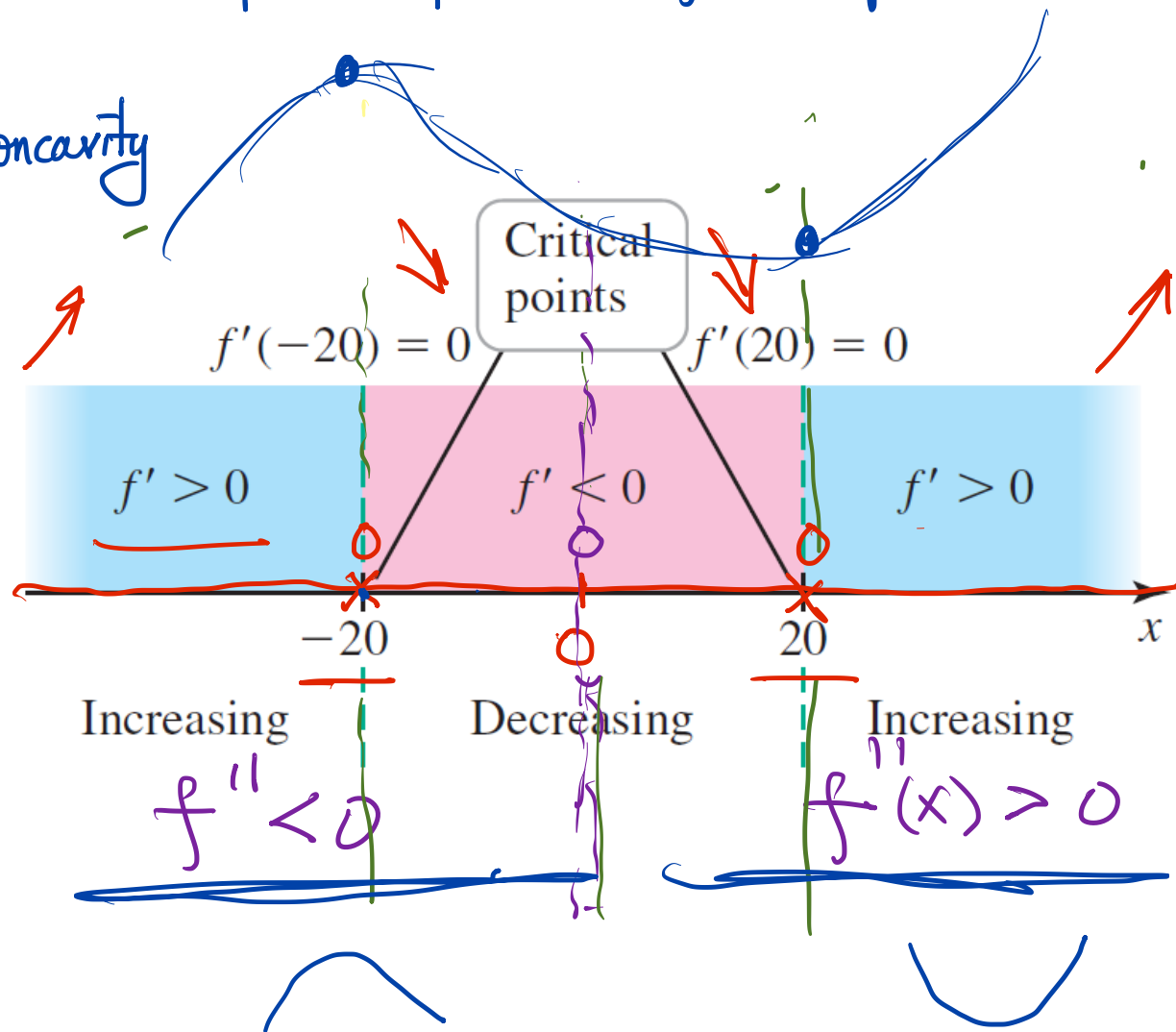
$$f''(x) = 2x = 0 \Rightarrow x = 0$$

$$= (x+20)(x-20) = 0 \Rightarrow x = -20, 20$$

Figure 4.47

(4) critical pts and possible inflection pts

(5) increasing/decreasing and concavity



(6) extreme values and inflection pts

(7) asymptotes and end behavior

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

$$\lim_{x \rightarrow +\infty} f(x) = \infty$$

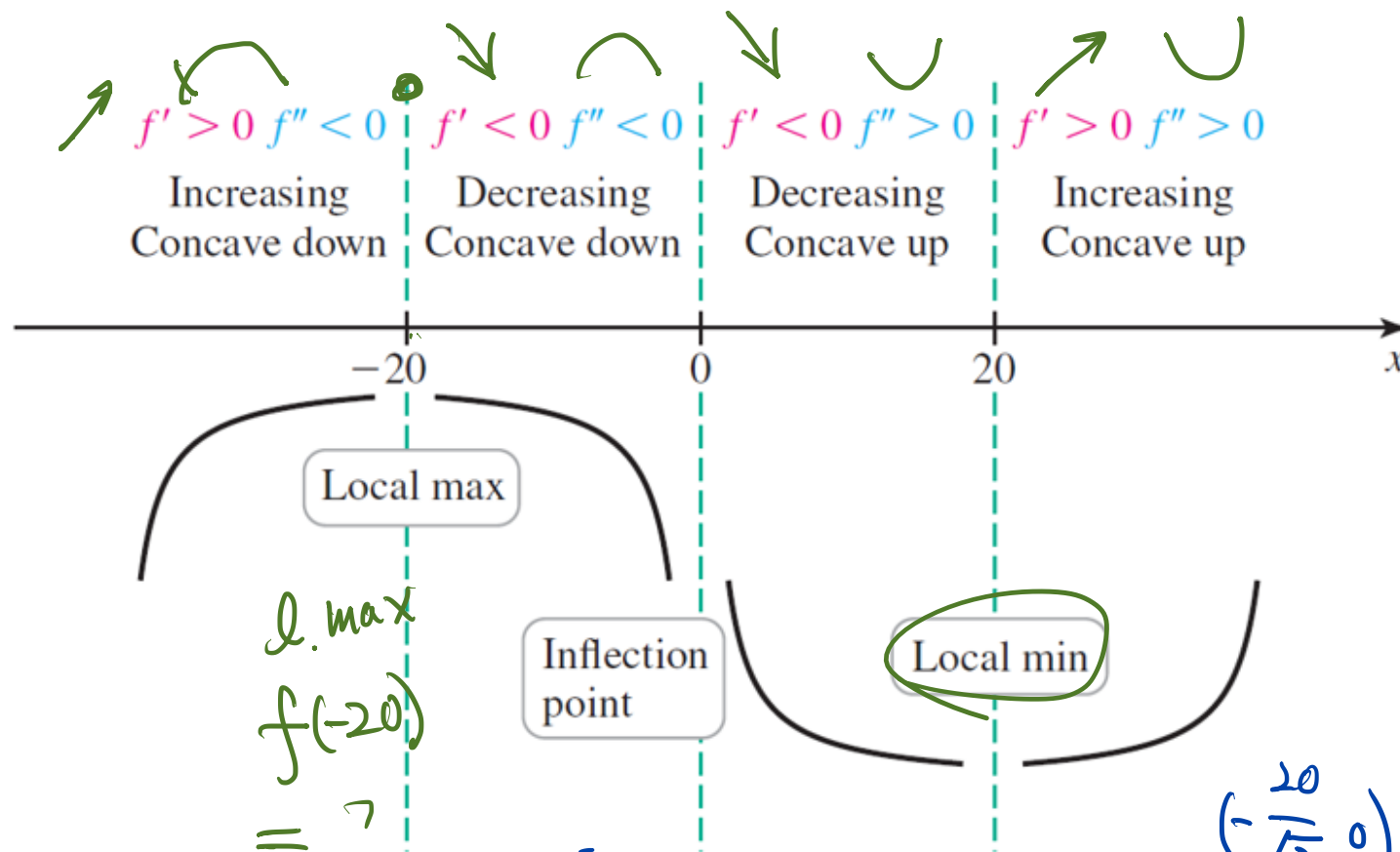
Figure 4.49

$$\frac{1}{3}x^3 - 400x$$

Behavior of f

$$-\infty \rightarrow \infty$$

$$\infty \rightarrow -\infty$$



Graph of f

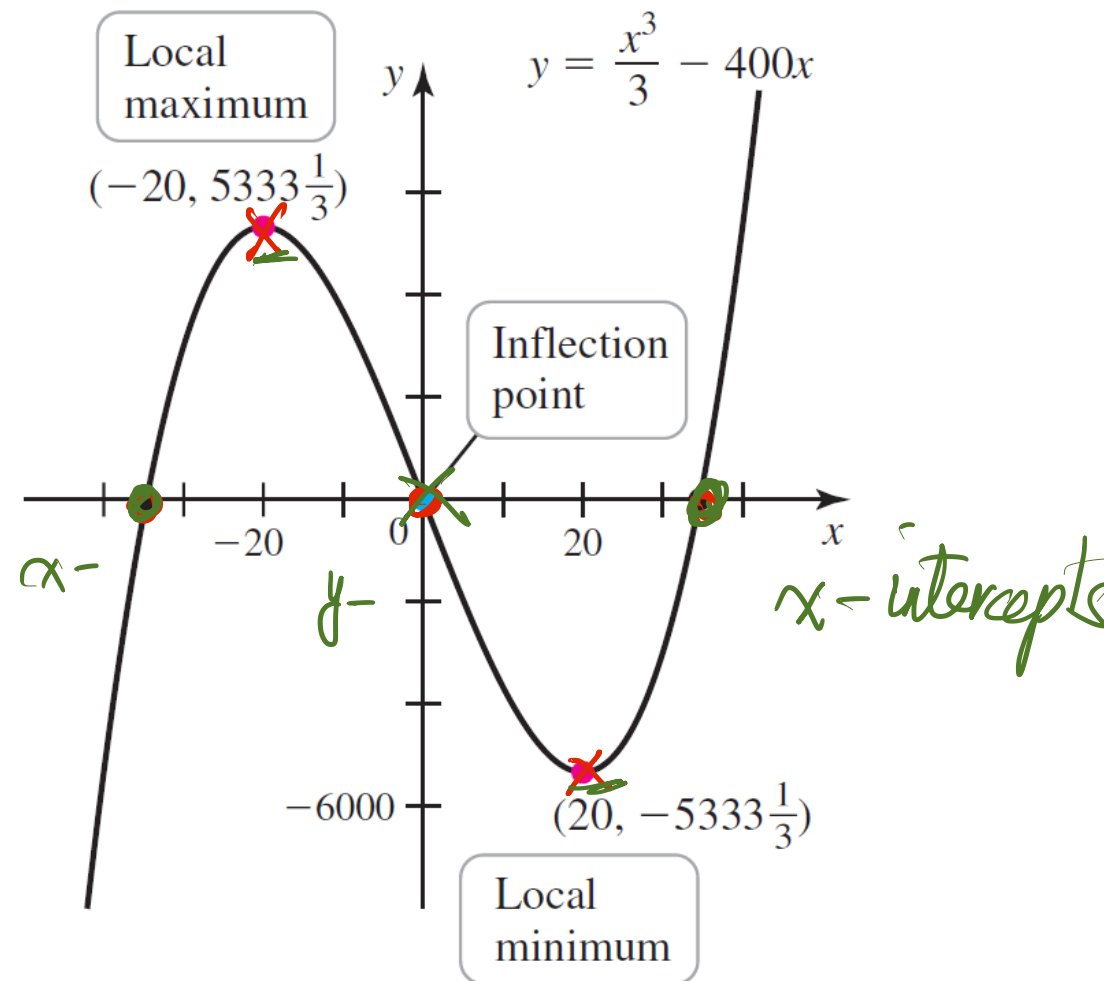
(8) intercepts

$$f(0) = 0 \text{ — y-intercept } (0, 0)$$

$$0 = f(x) = \frac{1}{3}x^3 - 400x = x \left[\frac{1}{3}x^2 - 400 \right]$$

$$x = \pm \frac{20}{\sqrt{3}}$$

Figure 4.50



$$x^2 - 1 = 0 \Rightarrow x = \pm 1$$



Example 3
 $f(x) = \frac{10x^3}{x^2 - 1}$

(1) domain $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ (2) symmetry $\lim_{x \rightarrow -1^-} f(x) = -\infty$
 (7) all asymptotes

Table 4.1

$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{10x^3}{x^2 - 1} = \lim_{x \rightarrow -\infty} \frac{10x}{1 - \frac{1}{x^2}} \rightarrow 10x = y$
 $\lim_{x \rightarrow 1^+} f(x)$

$\lim_{x \rightarrow +\infty} f(x)$	$20x$	$x^2 + 3$	$(x^2 - 1)^3$	Sign of f''
$(-\infty, -1)$	-	+	+	-
$(-1, 0)$	-	+	-	+
$(0, 1)$	+	+	-	-
$(1, \infty)$	+	+	+	+



(6) extreme values and inflection pts $\lim_{x \rightarrow -1^-} f(x) = -\infty$ $\lim_{x \rightarrow 1} f(x)$

(7) asymptotes and end behavior

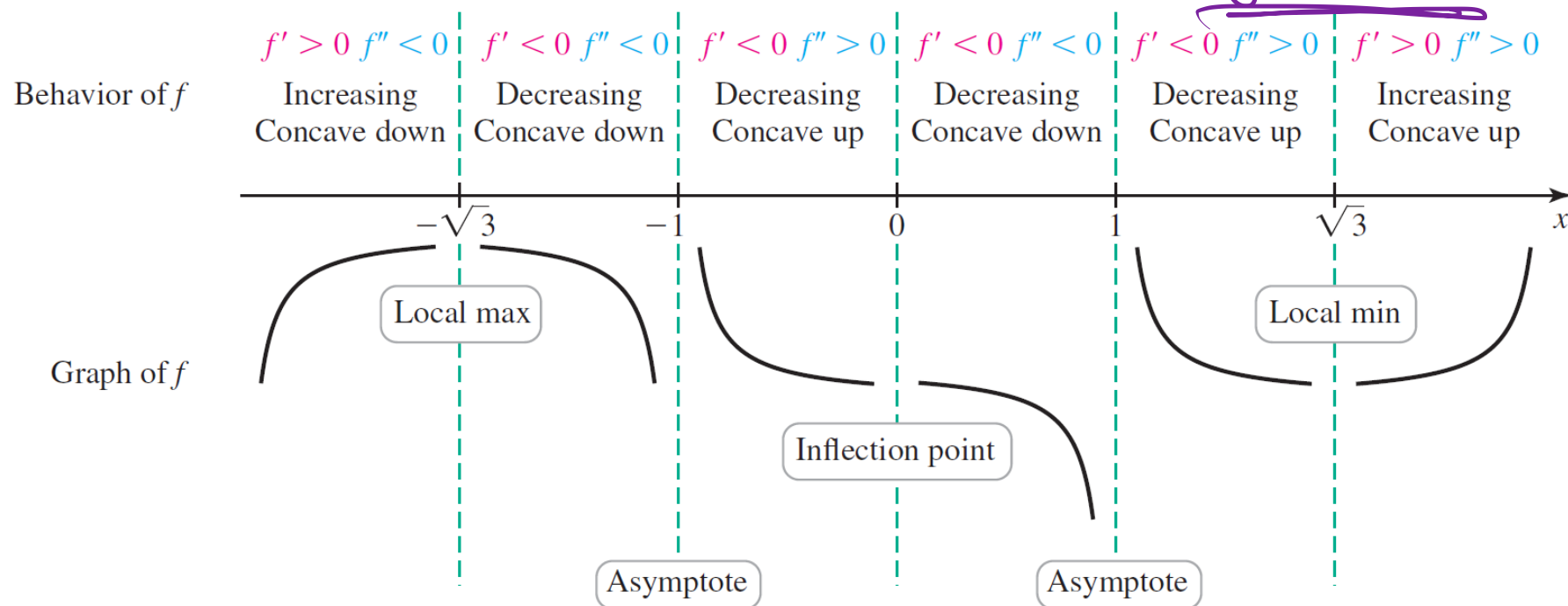
Figure 4.51

$$\lim_{x \rightarrow -\infty} \frac{10x^3}{x^2 - 1} = \infty$$

$x \rightarrow -1^-$ $x \rightarrow -1^+$

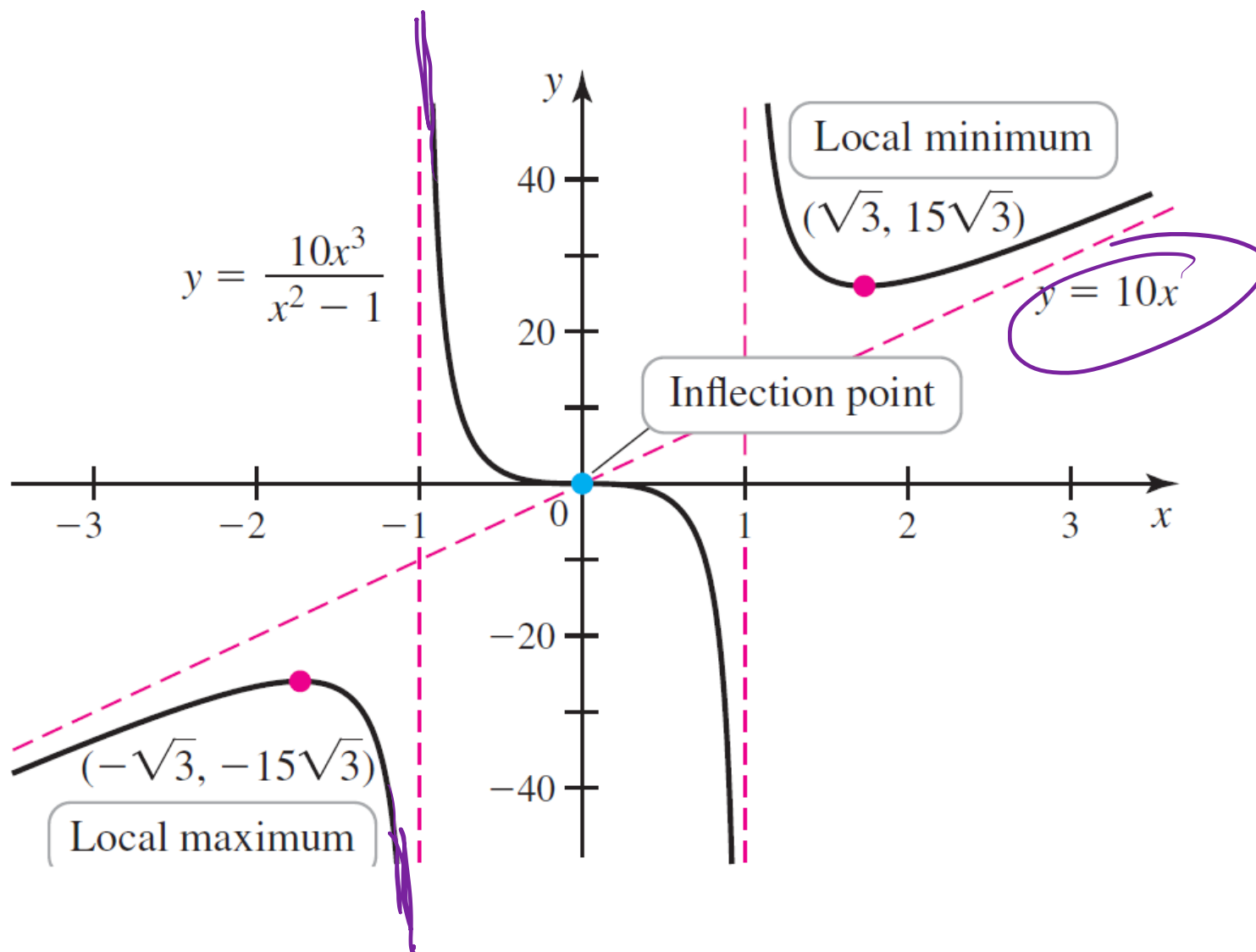
$$\lim_{x \rightarrow \infty} \frac{-10x^3}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{10x}{1 - \frac{1}{x^2}} \rightarrow 1$$

$y = 10x$



(8) intercepts

Figure 4.52



Example 4

$$f(x) = e^{-x^2}$$

Figure 4.53

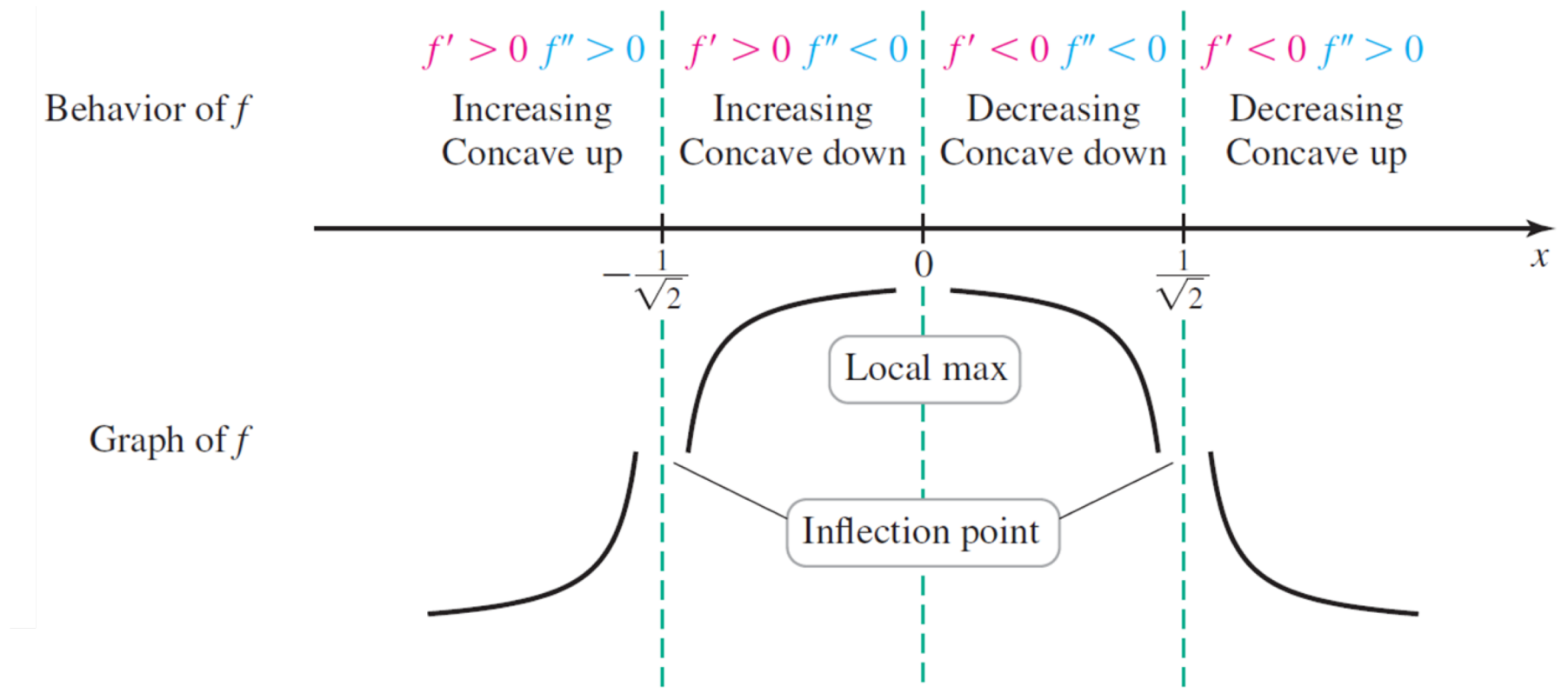
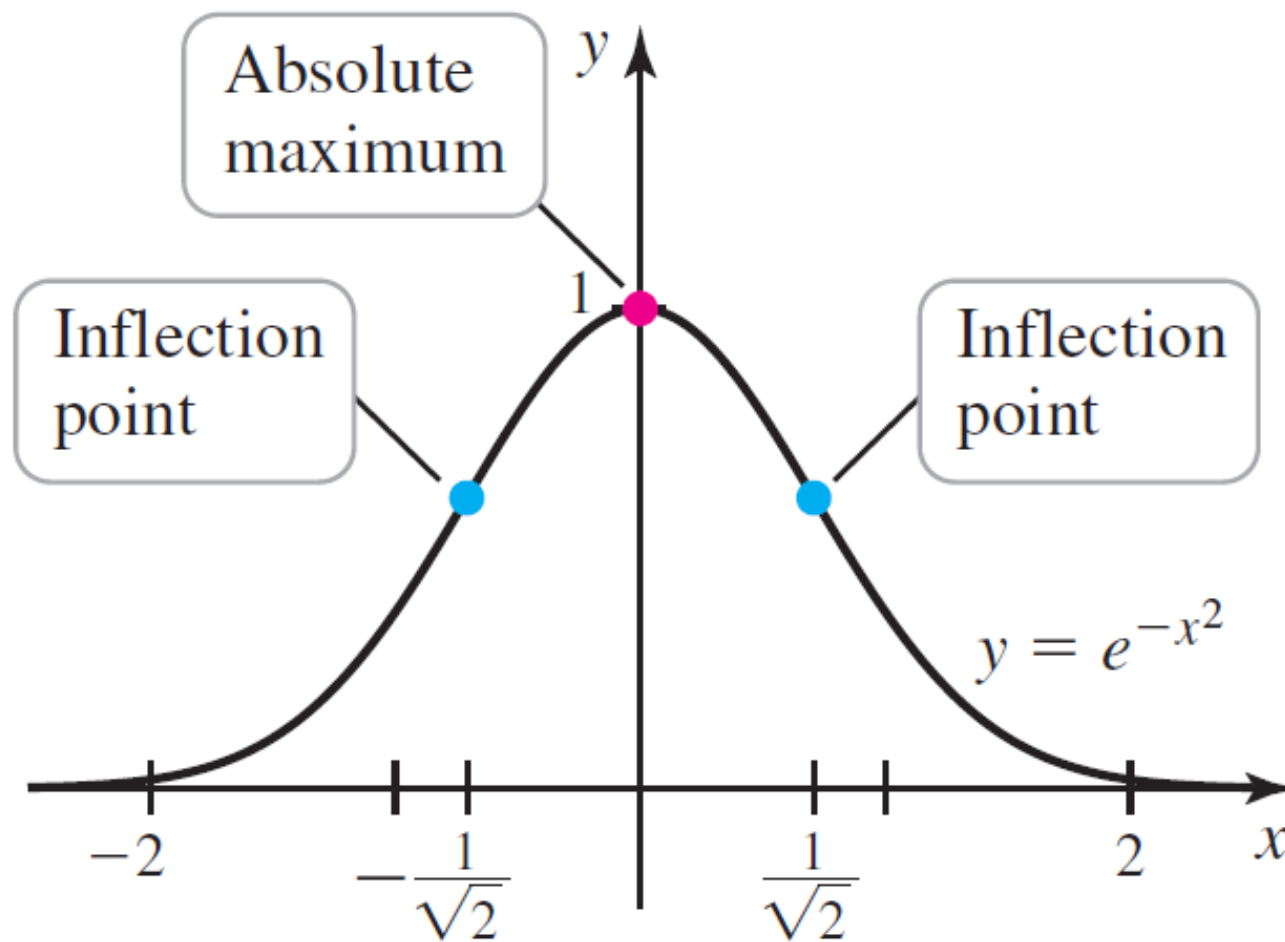


Figure 4.54



Example 5

$$f(x) = \frac{1}{8} x^{2/3} (9x^2 - 8x - 16)$$

Table 4.2

	$3x^{1/3}$	$9x + 4$	$x - 1$	Sign of f'
$(-\infty, -\frac{4}{9})$	—	—	—	—
$(-\frac{4}{9}, 0)$	—	+	—	+
$(0, 1)$	+	+	—	—
$(1, \infty)$	+	+	+	+

Figure 4.55

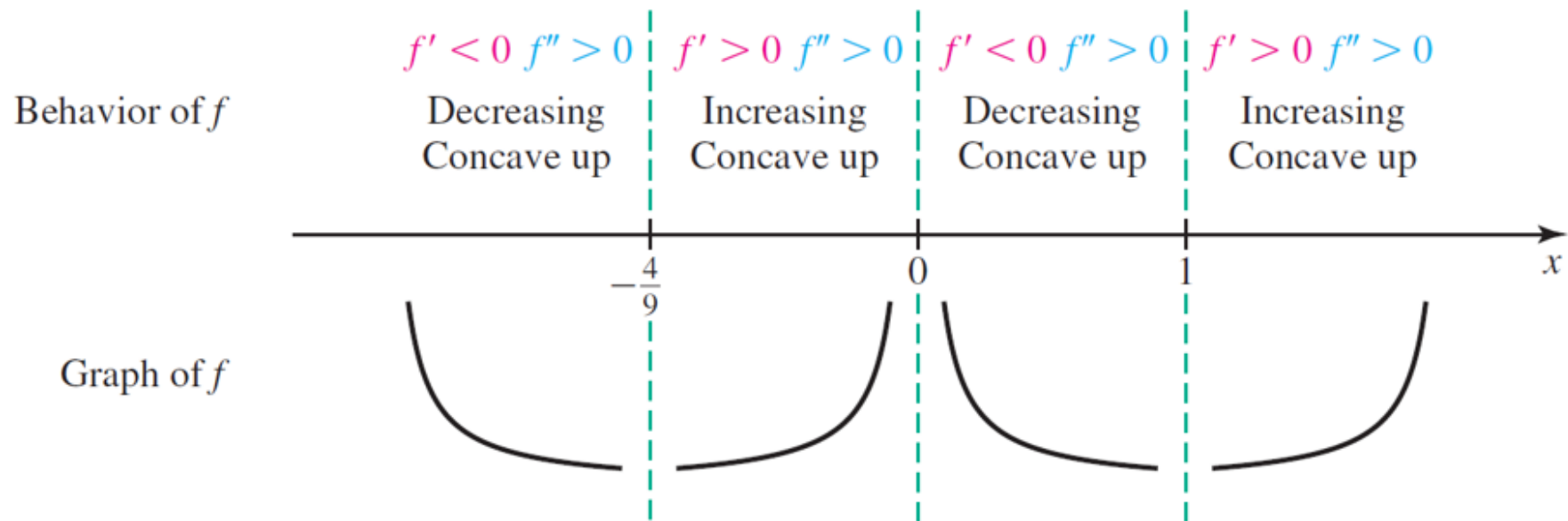
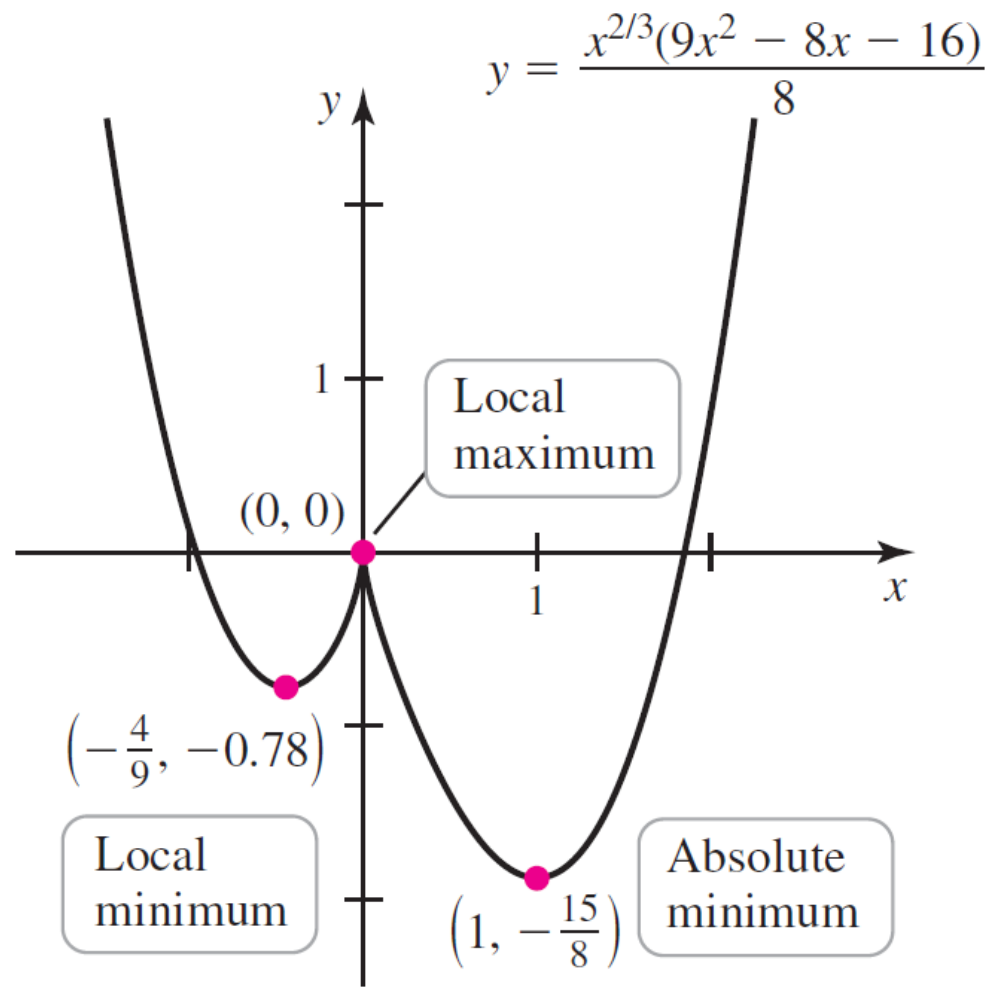


Figure 4.56



Section 4.5 Optimization Problems

Part I HW# 16, 21, 22, 27, 31, 32, 33, 34; Part II #37, 40, 45, ~~50~~, 51, 71, 77

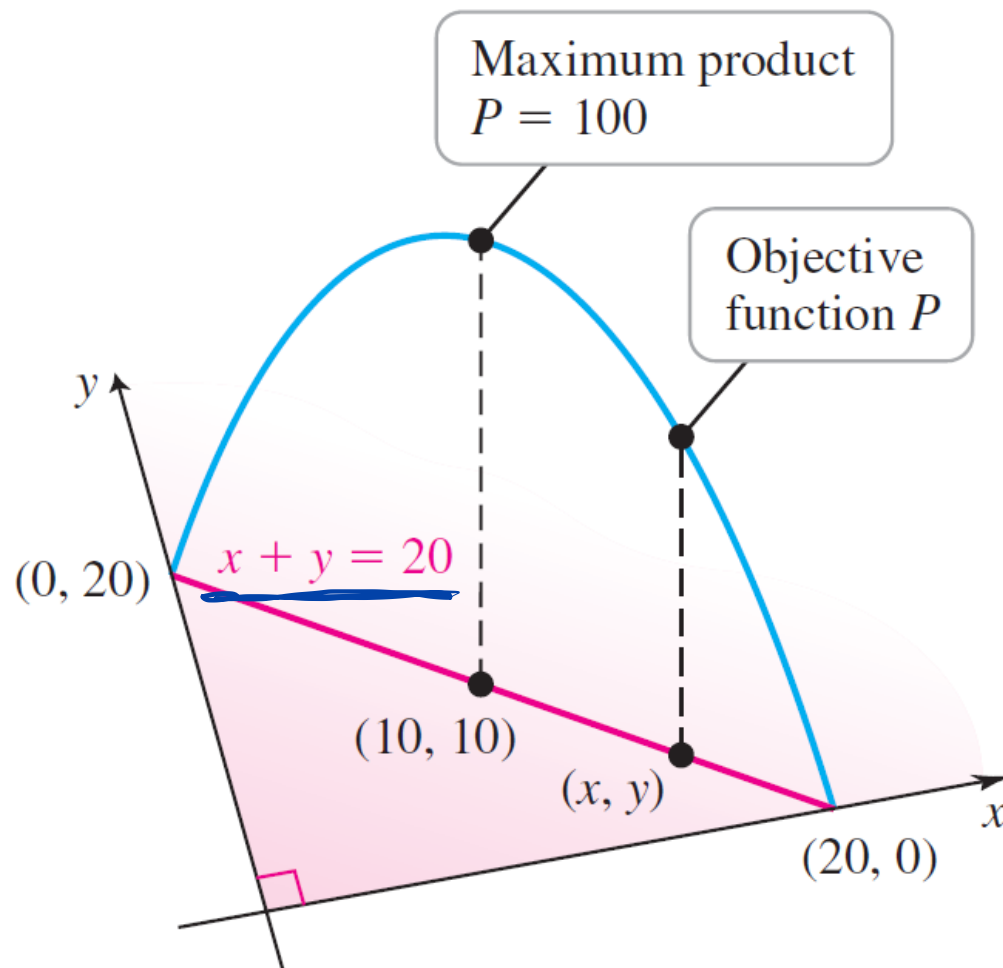
Example What pair of nonnegative numbers maximizes $P=xy$ subject to the constraint $x+y=20$? $\Rightarrow y=20-x$ $\parallel$ $x \in [0, 20]$

Table 4.3 $P'(x) = 20 - x + (-x) = 20 - 2x = 0 \Rightarrow x = 10$ $P(x) = x(20 - x)$ $P(0)=0, P(20)=0, P(10)=100$

x	y	$x + y$	$P = xy$
1	19	20	19
5.5	14.5	20	79.75
9	11	20	99
13	7	20	91
18	2	20	36

$P''(x) = -2 < 0$ 

Figure 4.57



Example 1 A rancher has 400 ft of fence for constructing a rectangular corral (see Fig. 4.58). What are the dimensions of the corral that maximize the enclosed area? What is the area of that corral?

Figure 4.58

$$A = xy \quad \text{s.t.} \quad 400 = 4x + y$$

$$x \in [0, 100]$$

$$y = 400 - 4x$$

$$A = x(400 - 4x)$$

$$A' = 400 - 4x + x(-4)$$

$$= 400 - 8x = 0$$

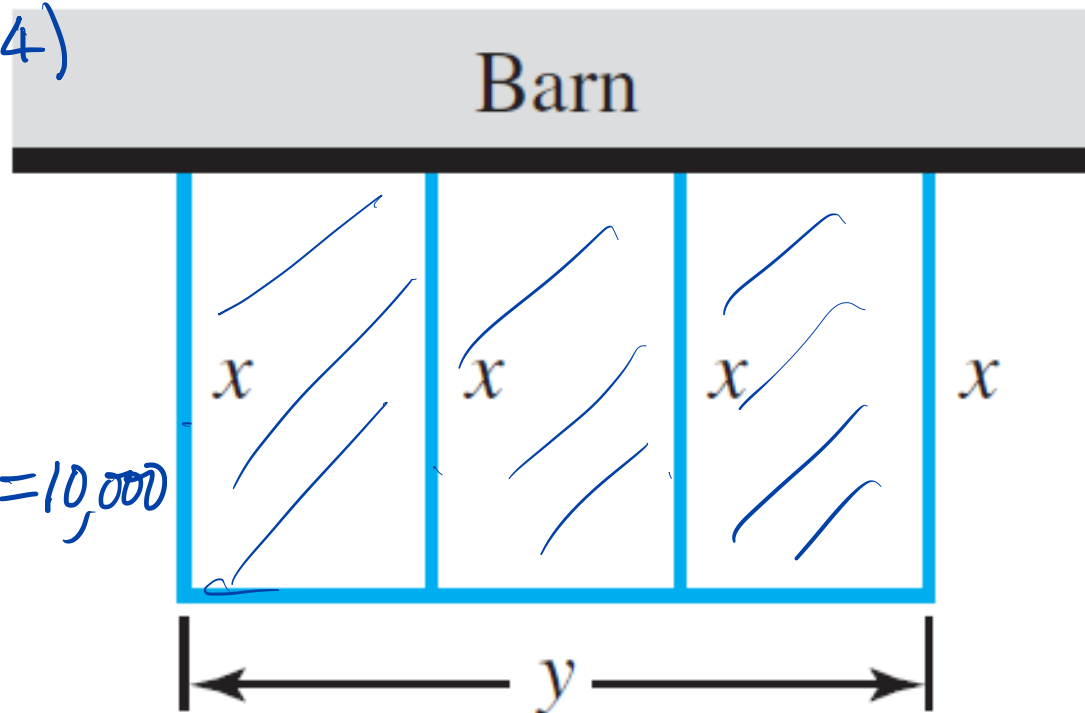
$$\Rightarrow x = \frac{400}{8} = 50$$

$$A'' = -8 < 0$$

$$A(50) = 50 \cdot 200 = 10,000$$

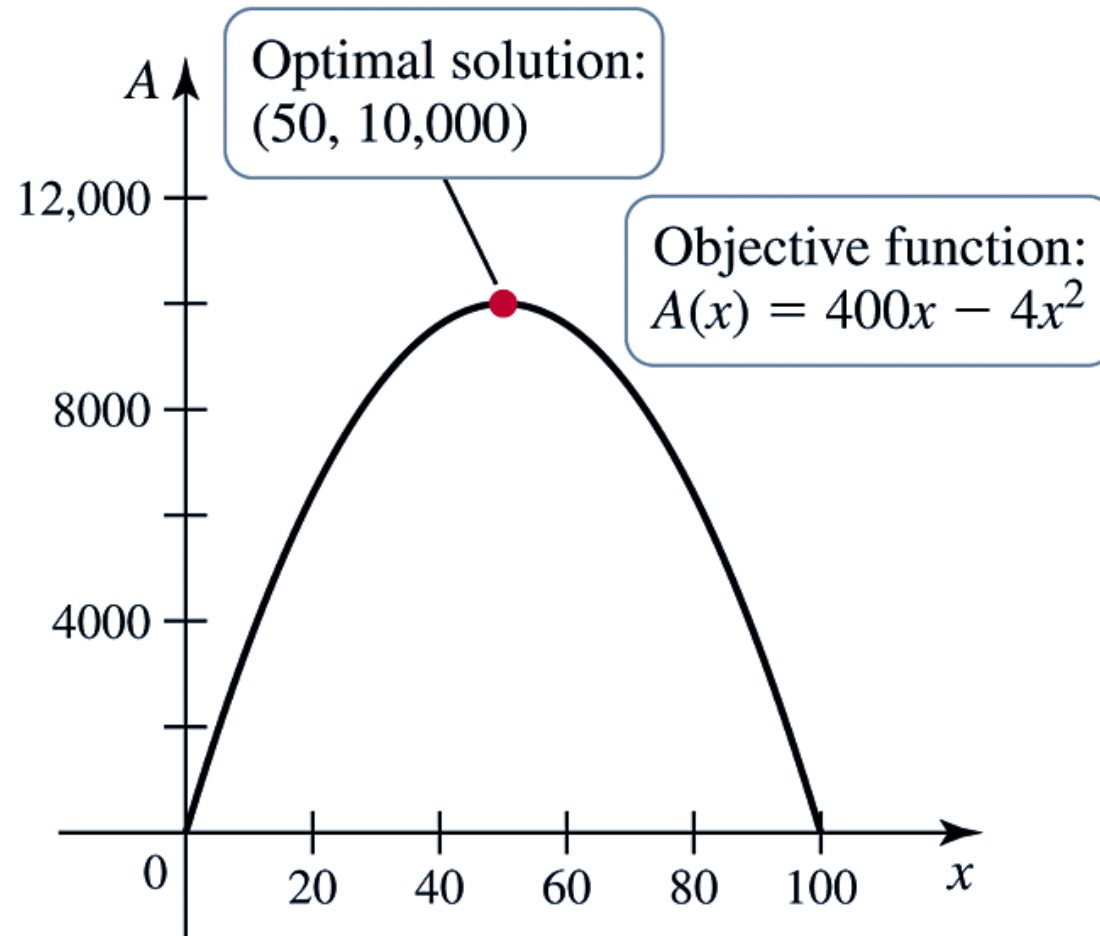
$$A(0) = 0$$

$$A(100) = 0$$



$$\text{Constraint: } 4x + y = 400$$

Figure 4.59



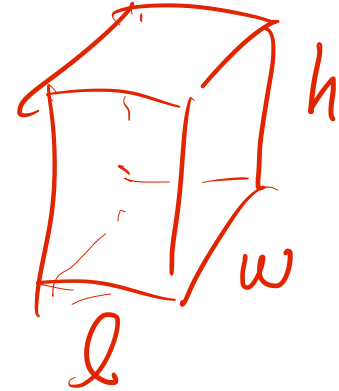
Example 2 Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width, and height not exceeding 64 in. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?

Figure 4.60

$$\max V = w^2 h \quad \text{s.t.} \quad 2w + h = 64$$

$\Downarrow$

$$h = 64 - 2w$$



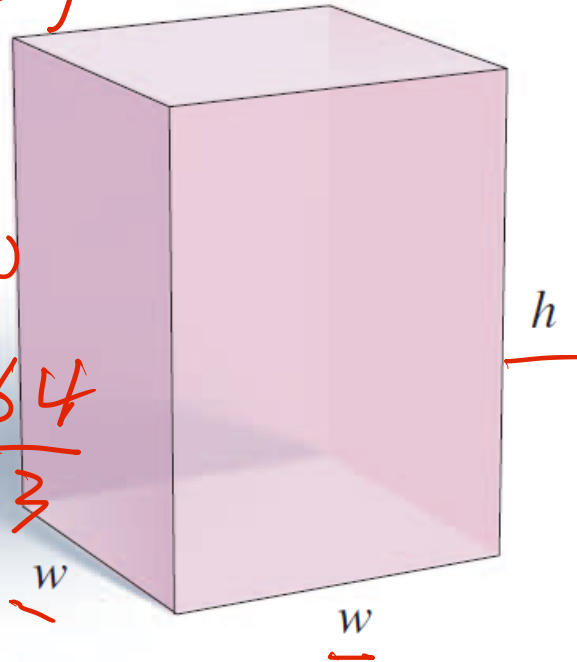
$$V = lwh$$

$$\Rightarrow V(w) = w^2 (64 - 2w)$$

$$\begin{aligned} V' &= [64w^2 - 2w^3]' \\ &= 128w - 6w^2 = (128 - 6w)w \\ &= 0 \end{aligned}$$

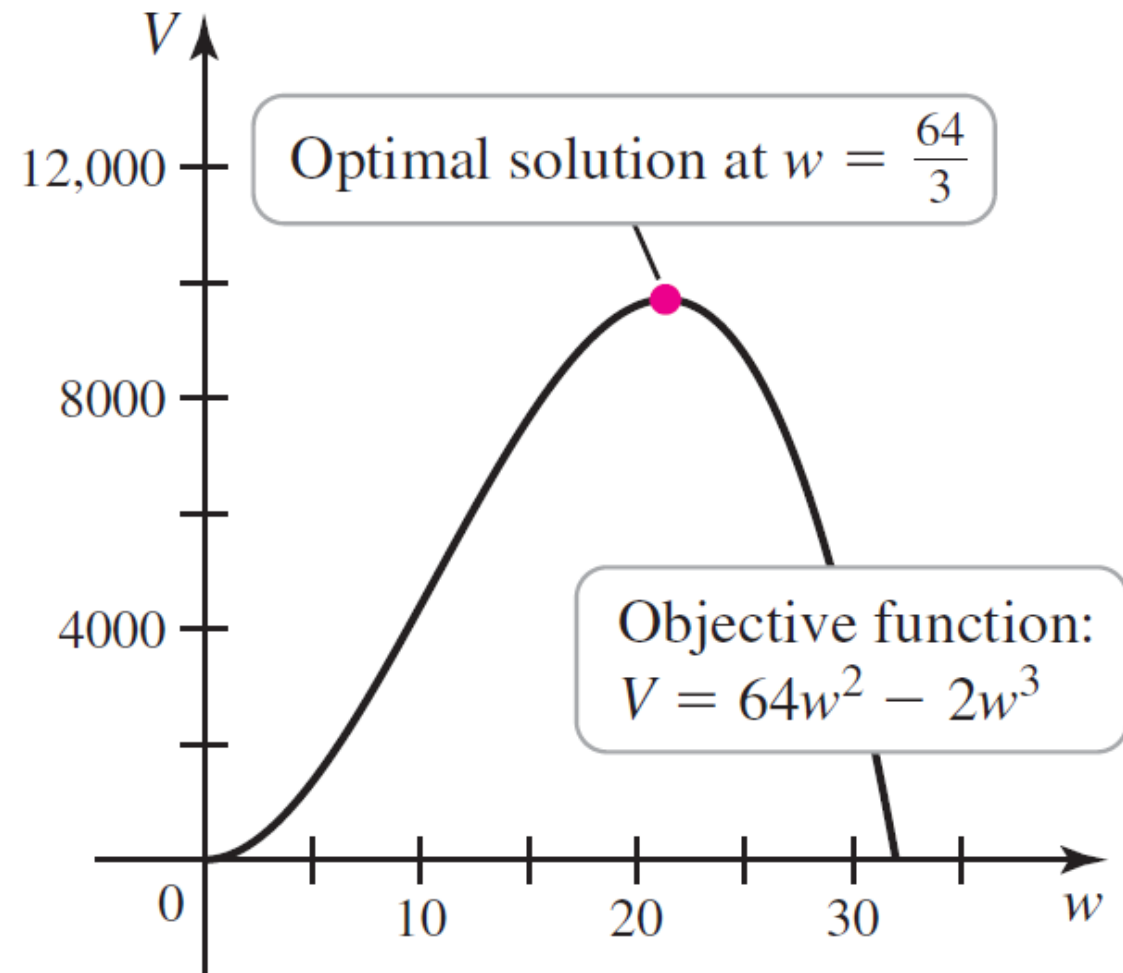
$$\Rightarrow w = 0 \text{ or } w = \frac{128}{6} = \frac{64}{3}$$

$$V(0) = 0; \quad V\left(\frac{64}{3}\right) =$$



Objective function: $V = w^2 h$
Constraint: $2w + h = 64$

Figure 4.61



Guidelines for Optimization Problems

1. Read the problem carefully, identify the variables, and organize the given information with a picture.
2. Identify the objective function (the function to be optimized). Write it in terms of the variables of the problem.
3. Identify the constraint(s). Write them in terms of the variables of the problem.
4. Use the constraint(s) to eliminate all but one independent variable of the objective function.
5. With the objective function expressed in terms of a single variable, find the interval of interest for that variable.
6. Use methods of calculus to find the absolute maximum or minimum value of the objective function on the interval of interest. If necessary, check the endpoints.

Example 3 Swim at 2 mi/hr, walk at 3 mi/hr. How to choose P to minimize the total time?

$$T = T_s + T_w = \frac{2 \sin \frac{\theta}{2}}{2} + \frac{(\pi - \theta) \cdot 1}{3}$$

$$= \sin \frac{\theta}{2} + \frac{1}{3} (\pi - \theta)$$

Figure 4.62

$$T'(\theta) = \frac{1}{2} \cos \frac{\theta}{2} - \frac{1}{3} = 0 \Rightarrow \cos \frac{\theta}{2} = \frac{2}{3} \Rightarrow \theta = 2 \cos^{-1} \left(\frac{2}{3} \right)$$

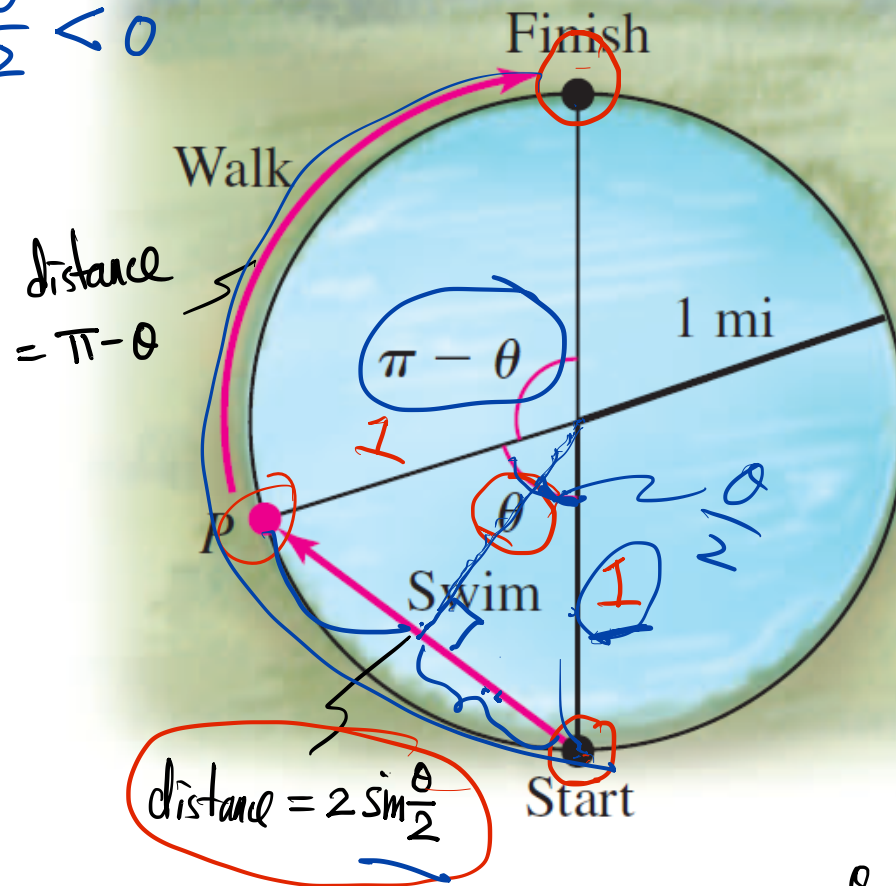
$$T''(\theta) = -\frac{1}{4} \sin \frac{\theta}{2} < 0$$

$$\theta \in [0, \pi]$$

$$\frac{\theta}{2} \in \left[0, \frac{\pi}{2} \right]$$

1 miles

$$\frac{\pi}{3}$$



$$T(\theta) = T(96^\circ)$$

$$T = 1.23 \text{ hr.}$$

is not a 1 min

$$T(0) = \pi/3 \approx 1.05 \text{ hr}$$

$$T(1.68) \approx 1.23 \text{ hr}$$

$$T(\pi) = 1 \text{ hr.}$$

$$\cos \frac{\theta}{2} = \frac{2}{3} \Rightarrow \theta \approx 1.68 \text{ rad} \approx 96^\circ$$

Figure 4.63

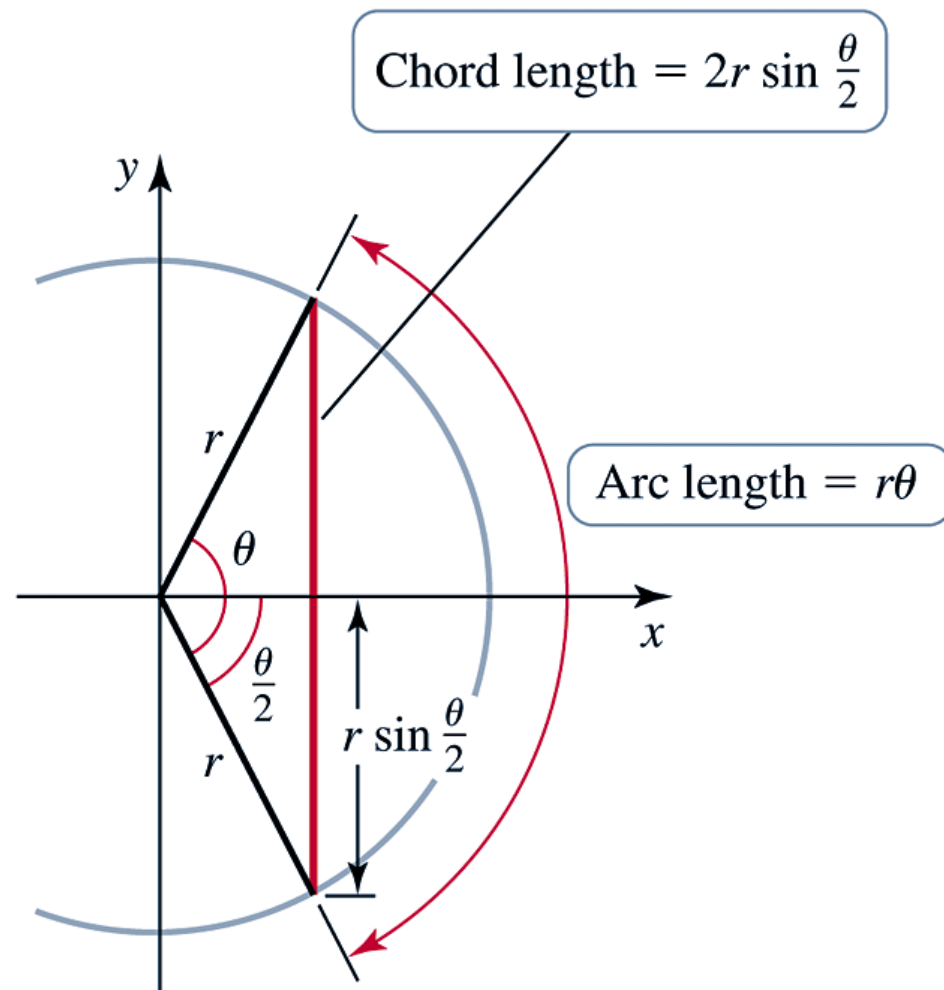
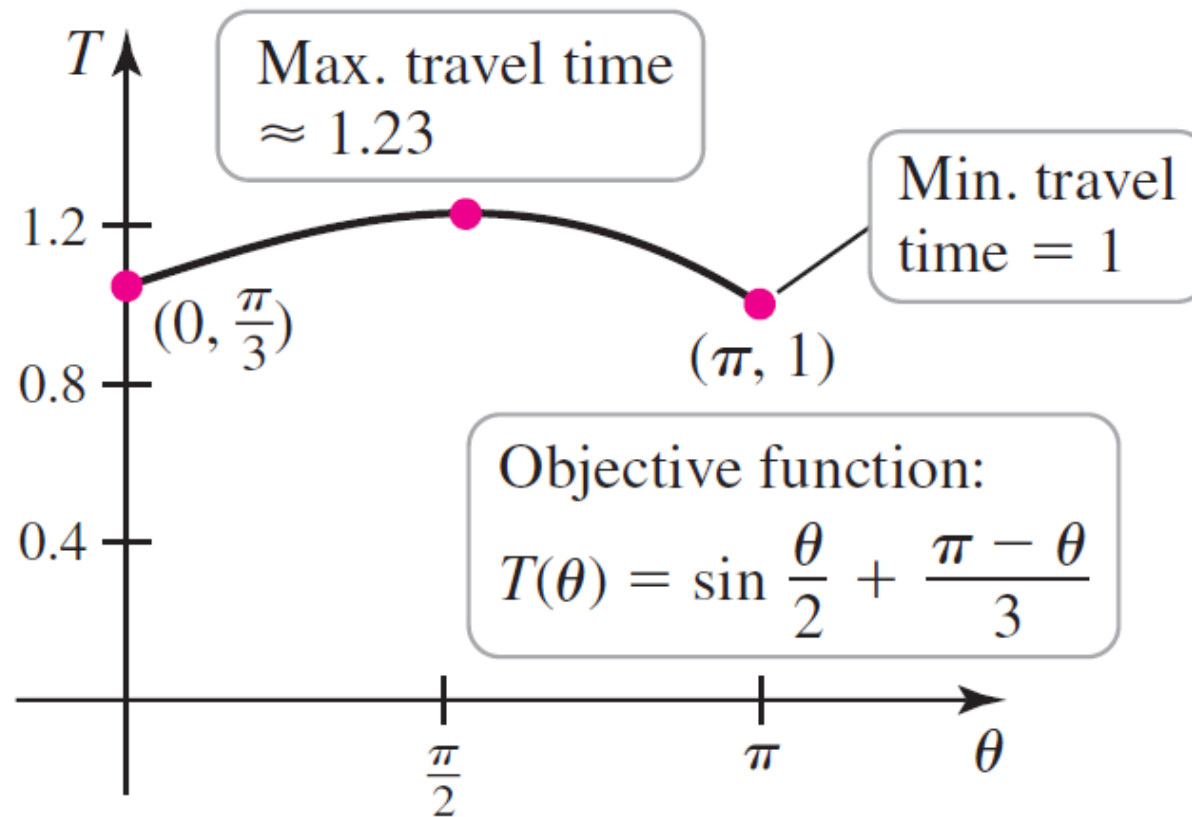
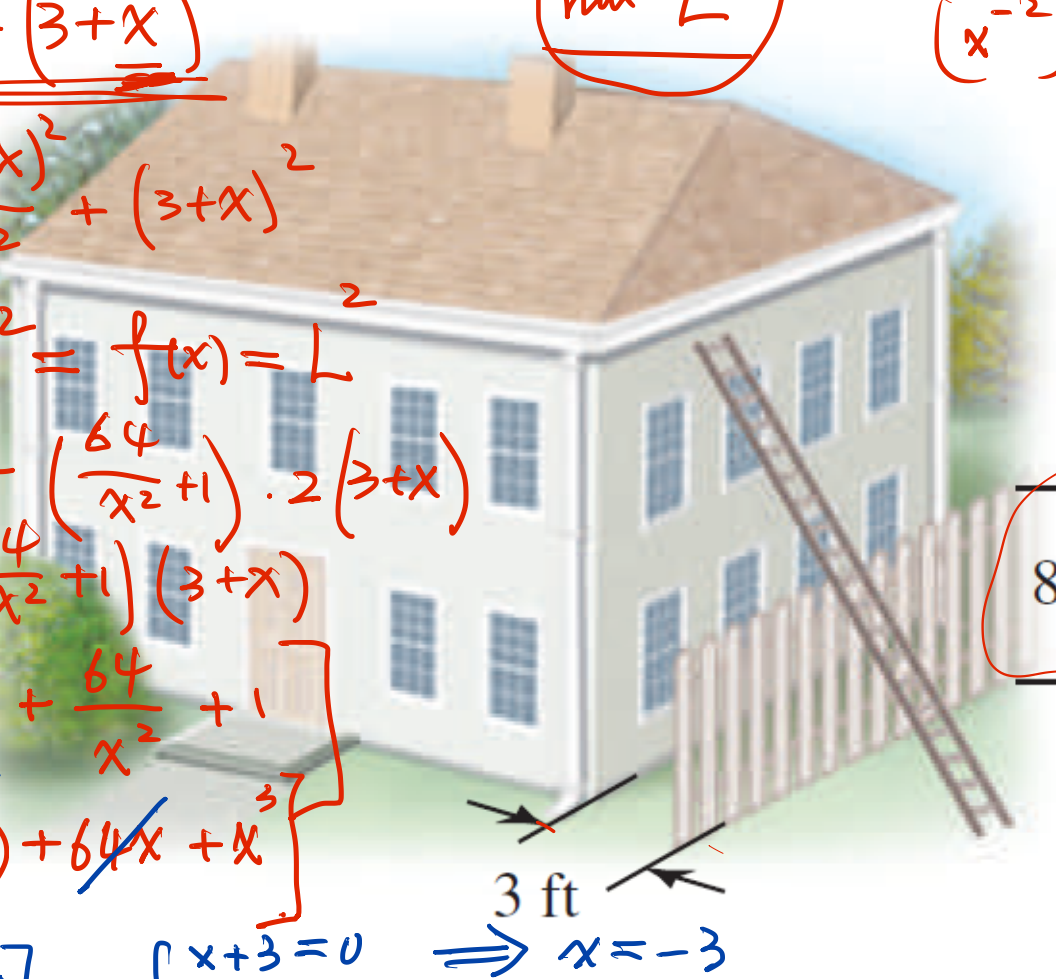


Figure 4.64



Example 4 An 8-foot-tall fence runs parallel to the side of a house 3 ft away. What is the length of the shortest ladder that clears the fence and reaches the house?

Figure 4.65 (a)



$$L^2 = 6^2 + (3+x)^2$$

$$= 64 \frac{(3+x)^2}{x^2} + (3+x)^2$$

$$= \left(\frac{64}{x^2} + 1 \right) (3+x)^2 = f(x) = L^2$$

$$f' = 64 \cdot \left(\frac{1}{x^2} \right)' (3+x)^2 + \left(\frac{64}{x^2} + 1 \right) \cdot 2(3+x)$$

$$= -128x^{-3} (3+x)^2 + \left(\frac{64}{x^2} + 1 \right) (3+x)$$

$$= 2(3+x) \left[-64 \frac{3+x}{x^3} + \frac{64}{x^2} + 1 \right]$$

$$= \frac{2(3+x)}{x^3} \left[-64(3+x) + 64x + x^3 \right]$$

$$= \frac{2(3+x)}{x^3} \left[x^3 - 3 \cdot 64 \right] \Rightarrow \begin{cases} x+3=0 \\ \text{or } x^3 - 192=0 \end{cases} \Rightarrow x = -3$$

$$(x^{-2})' = -2x^{-3}$$

$$\min L$$

8 ft

3 ft

x

L

3

8

x

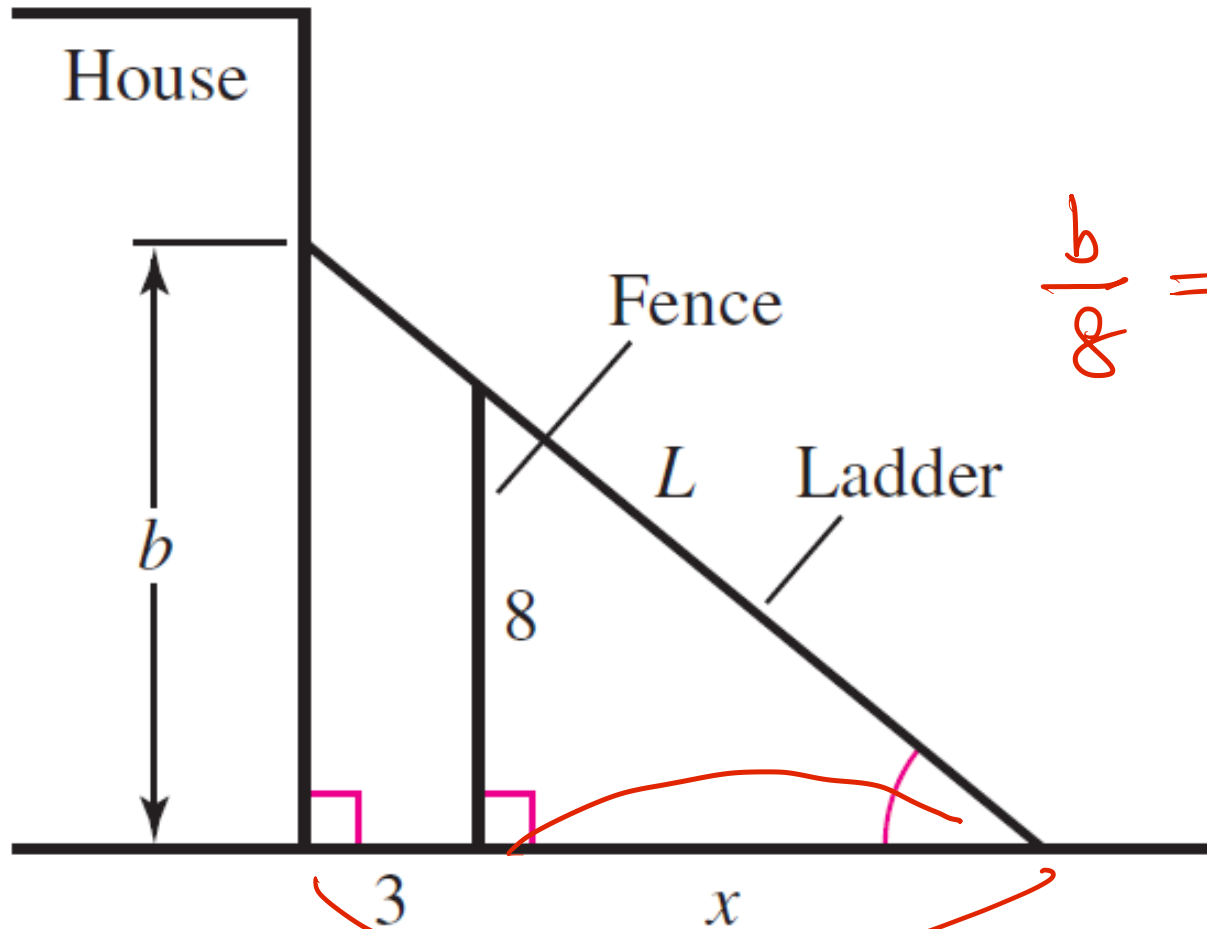
$$0 = x^3 - \boxed{192} = (x - \sqrt[3]{192}) \left(x^2 + \sqrt[3]{192} x + 192^{2/3} \right) \Rightarrow x = \sqrt[3]{192}$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Figure 4.65 (b)

$$a^3 + a^2b + ab^2 - ba^2 - ab^2 - b^3$$

$$L^2 = f\left(\sqrt[3]{192}\right)$$



$$\frac{b}{8} = \frac{3+x}{x}$$

Example 5 A water storage tank in a small community is built in the shape of a right circular cylinder with a capacity of $32,000 \text{ ft}^3$ (about 240,000 gallons). The interior wall and floor of the tank must be cleaned and treated annually. Labor costs for cleaning the wall are twice as high per square foot as the cost to clean the floor. Find the radius and height of the tank that minimizes the cleaning cost.

Figure 4.66

Given

$$V = 32,000 = \pi r^2 h$$

↓

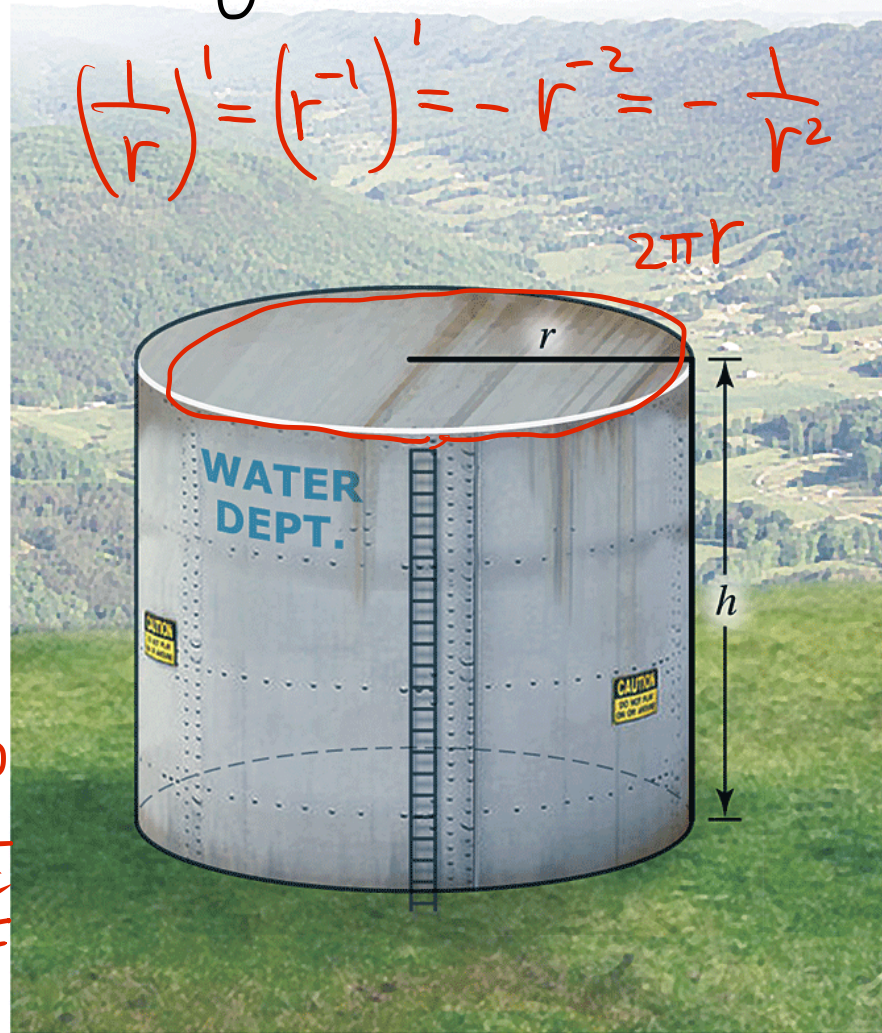
$$h = \frac{32,000}{\pi r^2}$$

$$f(r) = \pi r^2 + \frac{128}{r}$$

$$f'(r) = 2\pi r - \frac{128}{r^2}$$

$$= \frac{2}{r^2} [\pi r^3 - 64] = 0$$

$$r^3 = \frac{64}{\pi} \Rightarrow r = \sqrt[3]{\frac{64}{\pi}}$$



$$\left(\frac{1}{r}\right)' = (r^{-1})' = -r^{-2} = -\frac{1}{r^2}$$

min C

$$\begin{aligned} C &= C_i + C_f \\ &= 2m(2\pi r)h + m\pi r^2 \\ &= m\pi [4rh + r^2] \\ &= m\pi \left[\frac{128r}{\pi r^2} + r^2 \right] \\ &= m \left[\frac{128}{r} + \pi r^2 \right] \end{aligned}$$

Section 4.6 Linear Approximation and Differentials

HW # 3, 4, 7, 25, 29, 30, 33, 13, 15, 38, 39, 41, 43, 53, 56, 59, 63, 69

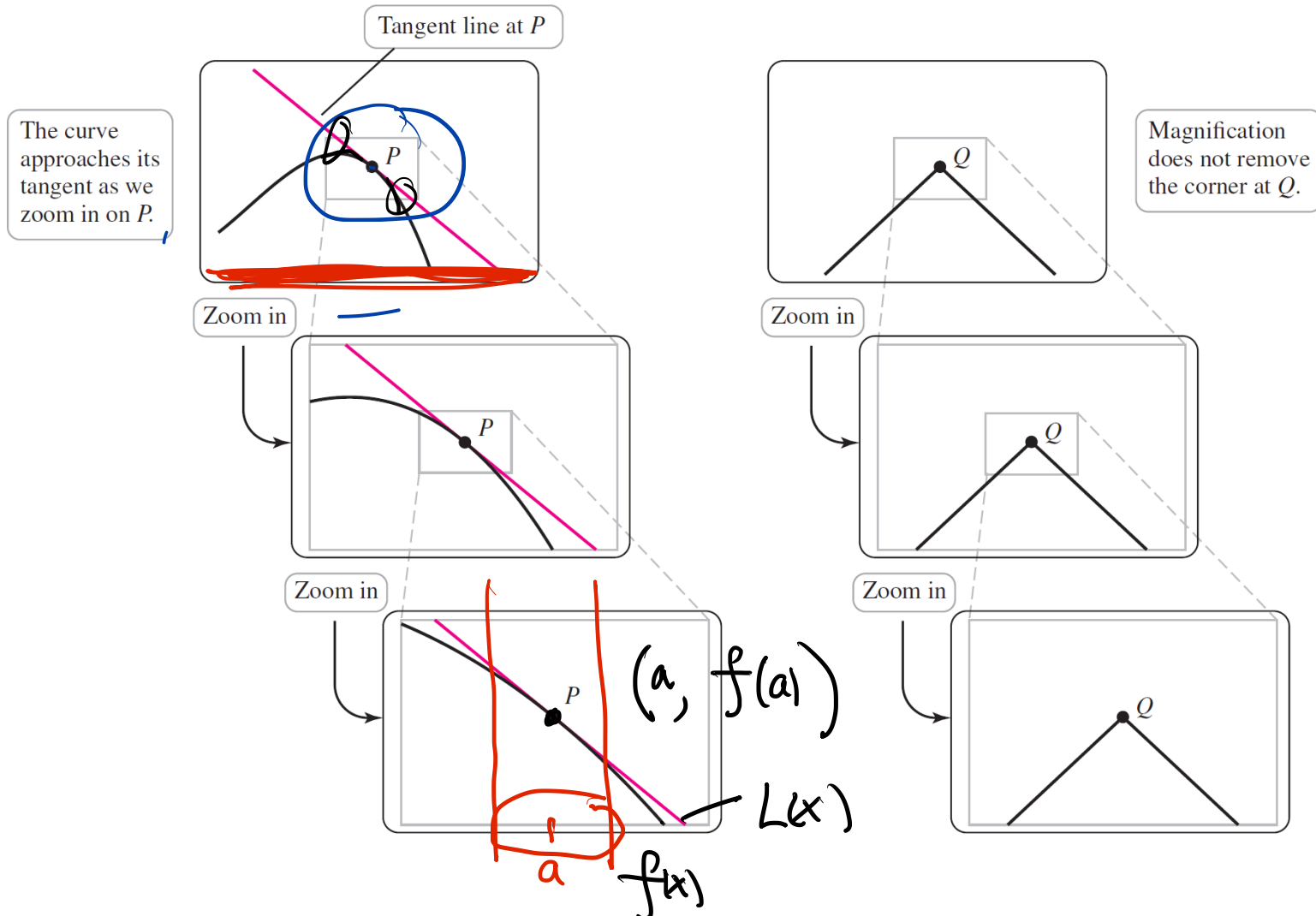
$$\underbrace{f(x)} \approx \underbrace{f(a)} + \underbrace{f'(a)}(x-a) = \underbrace{L(x)}$$

at $x=a=?$

$$\begin{array}{l} L(a) = f(a) \\ \hline L'(a) = f'(a) \end{array}$$

$$m = \frac{y-b}{x-a}$$

Figure 4.67 (a & b)



• Linear Approximation to $f(x)$ at $x=a$

$$f(x) \approx L(x) = f(a) + f'(a)(x-a)$$

Figure 4.68

Examples

(1) $f(x) = \sqrt{x}$ at $x=1$ and $x=4$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$L_1(x) = f(1) + f'(1)(x-1) = 1 + \frac{1}{2}(x-1) = \frac{1}{2}x + \frac{1}{2}$$

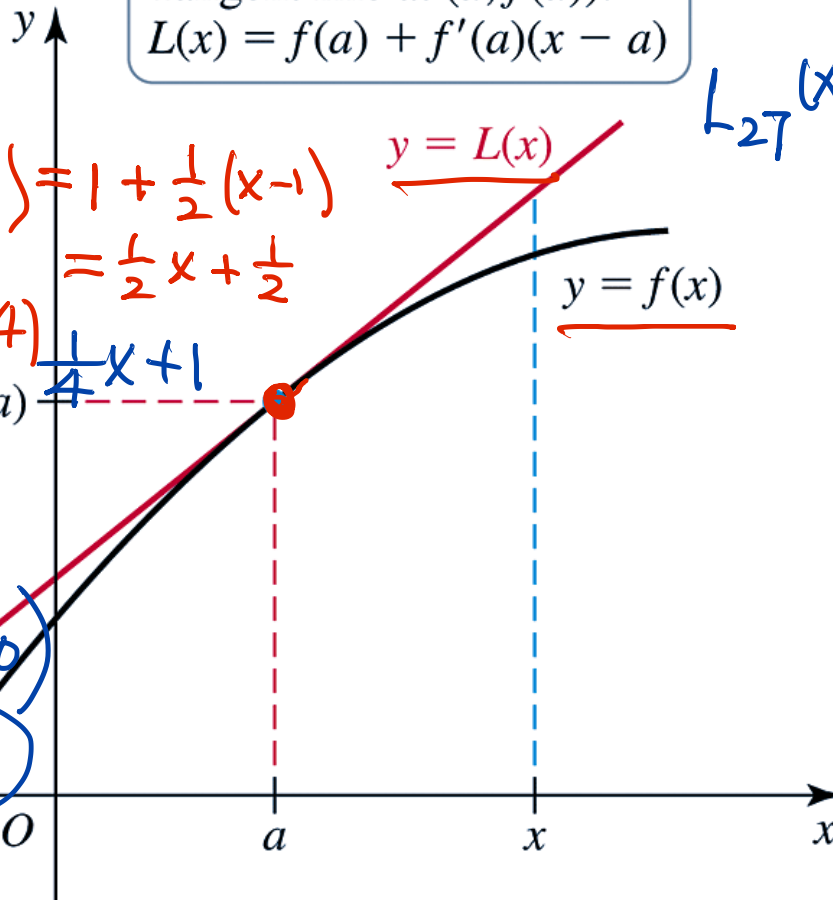
$$L_4(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4) = \frac{1}{4}x + 1$$

(2) $f(x) = \sin x$ at $x=0, \frac{\pi}{2}$

$$f'(x) = \cos x$$

$$L_0(x) = f(0) + f'(0)(x-0) = 0 + (x-0) = x$$

Tangent line at $(a, f(a))$:
 $L(x) = f(a) + f'(a)(x-a)$



$$(3) f(x) = x^{\frac{1}{3}} \text{ at } x=1, 27=3^3$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$L_1(x) = f(1) + f'(1)(x-1) = 1 + \frac{1}{3}(x-1) = \frac{1}{3}x + \frac{2}{3}$$

$$L_{27}(x) = f(27) + f'(27)(x-27) = 3 + \frac{1}{9}(x-27) = \frac{1}{9}x$$

Example 2 Estimate $\sqrt{1.01}$ and $\sqrt{3.98}$

• $f(x) = \sqrt{x}$

$$f(1.01) \approx L_1(1.01) = f(1) + f'(1)(1.01 - 1)$$

$$= 1 + \left. \frac{1}{2} x^{-\frac{1}{2}} \right|_{x=1} 0.01$$

$$= 1 + \frac{1}{2} - 0.01 = 1.005$$

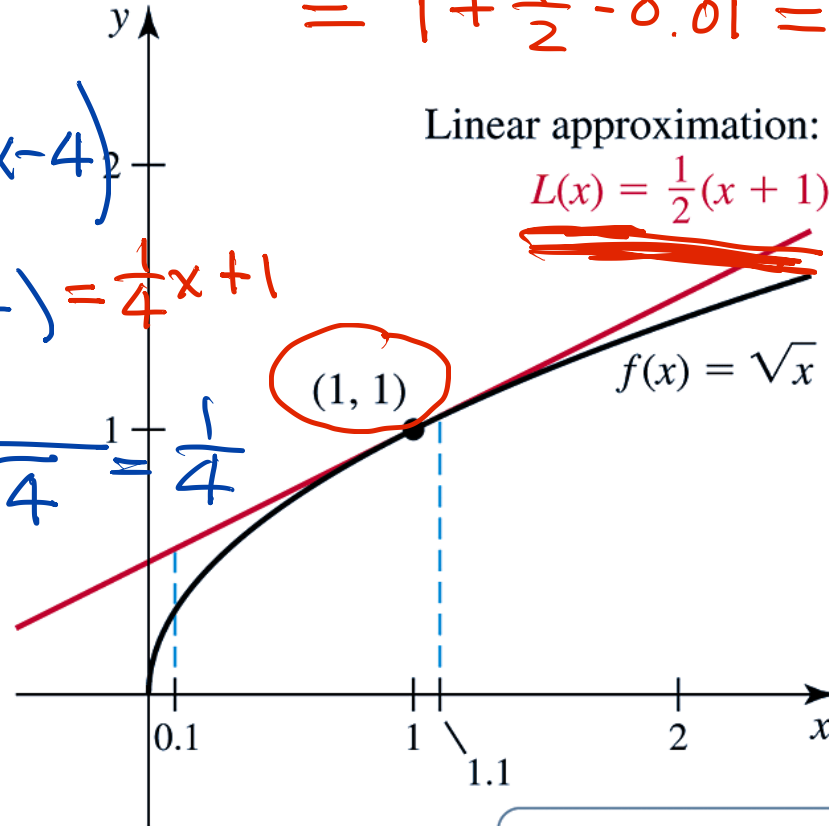
$a=?$ **Figure 4.69**

$$f(3.98) = \sqrt{3.98} \approx \frac{3.98}{4} + 1$$

$$f(x) \approx f(4) + f'(4)(x-4)$$

$$= 2 + \frac{1}{4}(x-4) = \frac{1}{4}x + 1$$

$$f'(4) = \left. \frac{1}{2} x^{-\frac{1}{2}} \right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$



$$L(1.1) = 1.05$$

$$\sqrt{1.1} = 1.0488\dots$$

The error in the approximation decreases as $x \rightarrow 1$.

Table 4.4

x	$L(x)$	Exact $\sqrt{x}$	Error
1.2	1.1	1.0954 ...	4.6×10^{-3}
1.1	1.05	1.0488 ...	1.2×10^{-3}
1.01	1.005	1.0049 ...	1.2×10^{-5}
1.001	1.0005	1.0005 ...	1.2×10^{-7}

Example Estimate $\sin 2.5^\circ$.

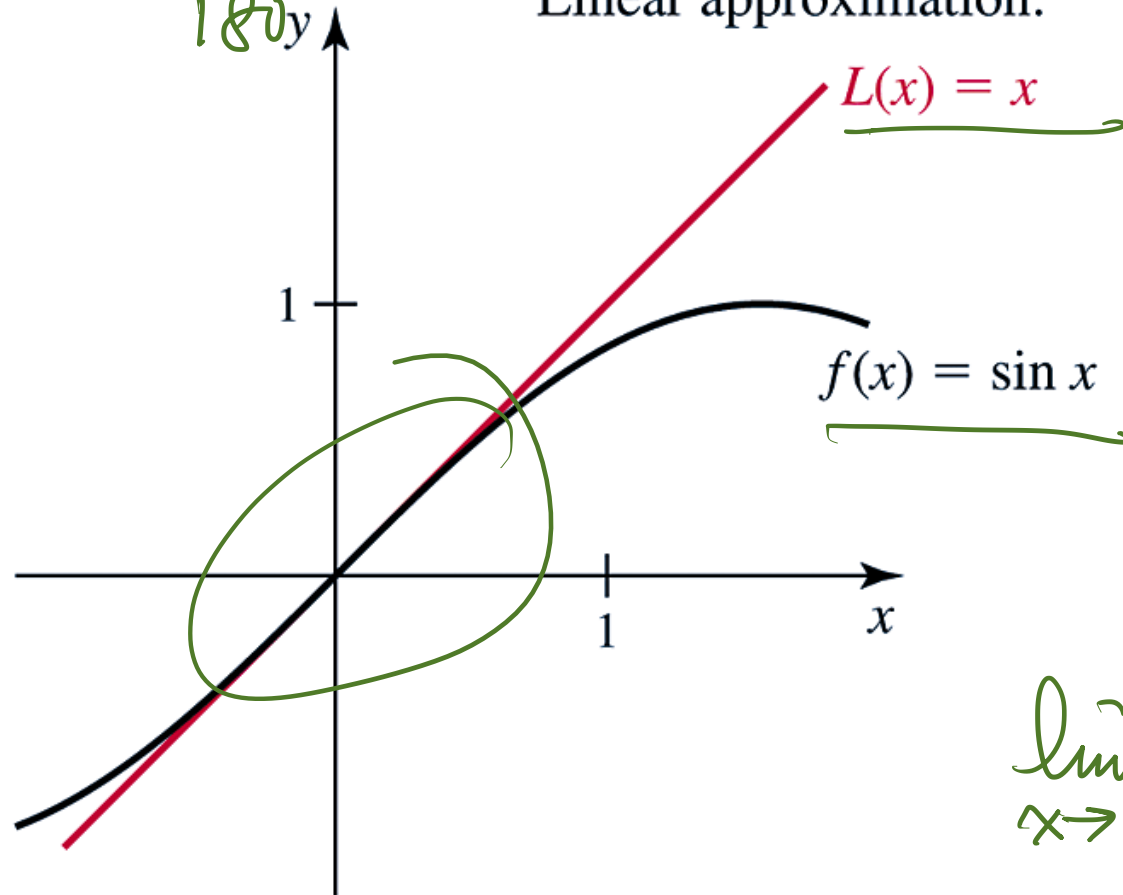
- $f(x) = \sin x$ $f' = \cos x$

- $a = 0$

- $f(x) \approx f(0) + f'(0)(x - 0)$
 $= x$

Linear approximation:

$$\sin 2.5^\circ \approx 2.5^\circ = \frac{2.5 \times \pi}{180}$$



$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

- over and under estimate

Figure 4.71 (a)

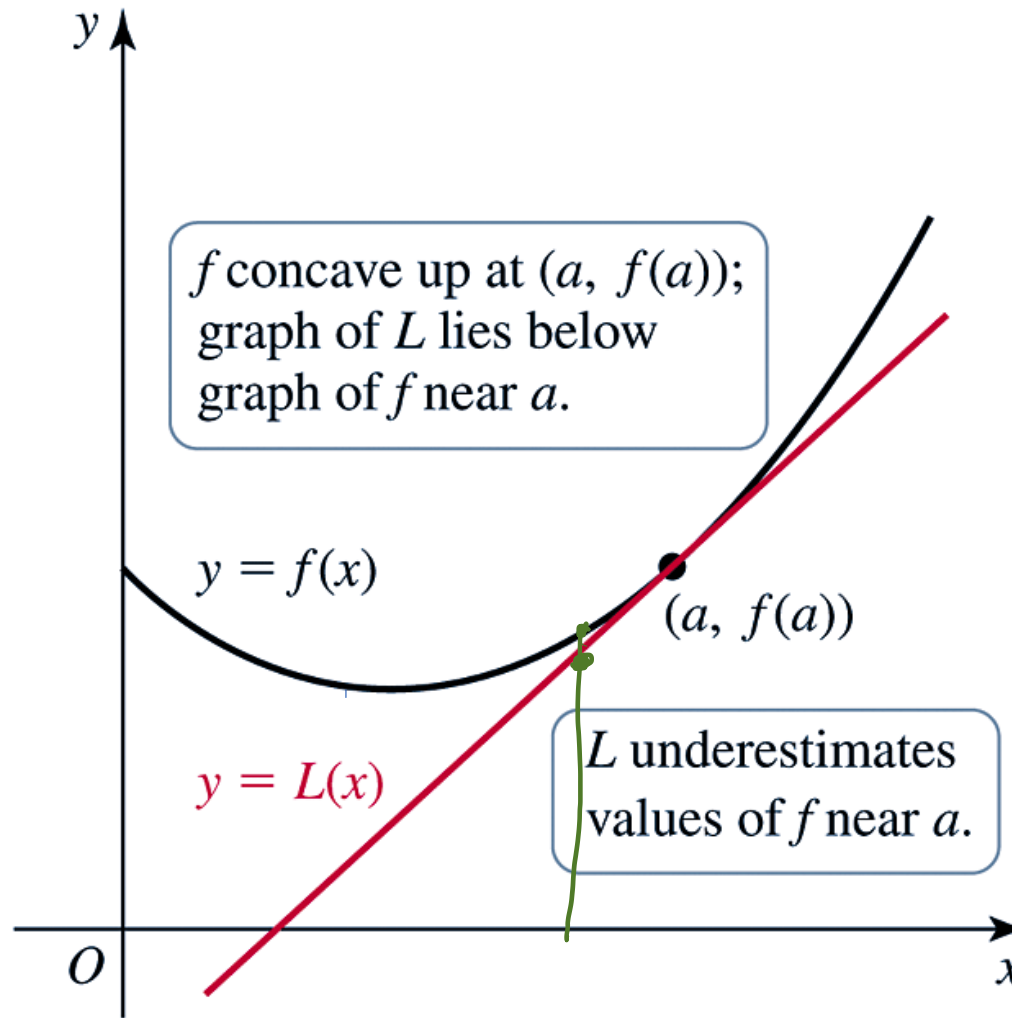
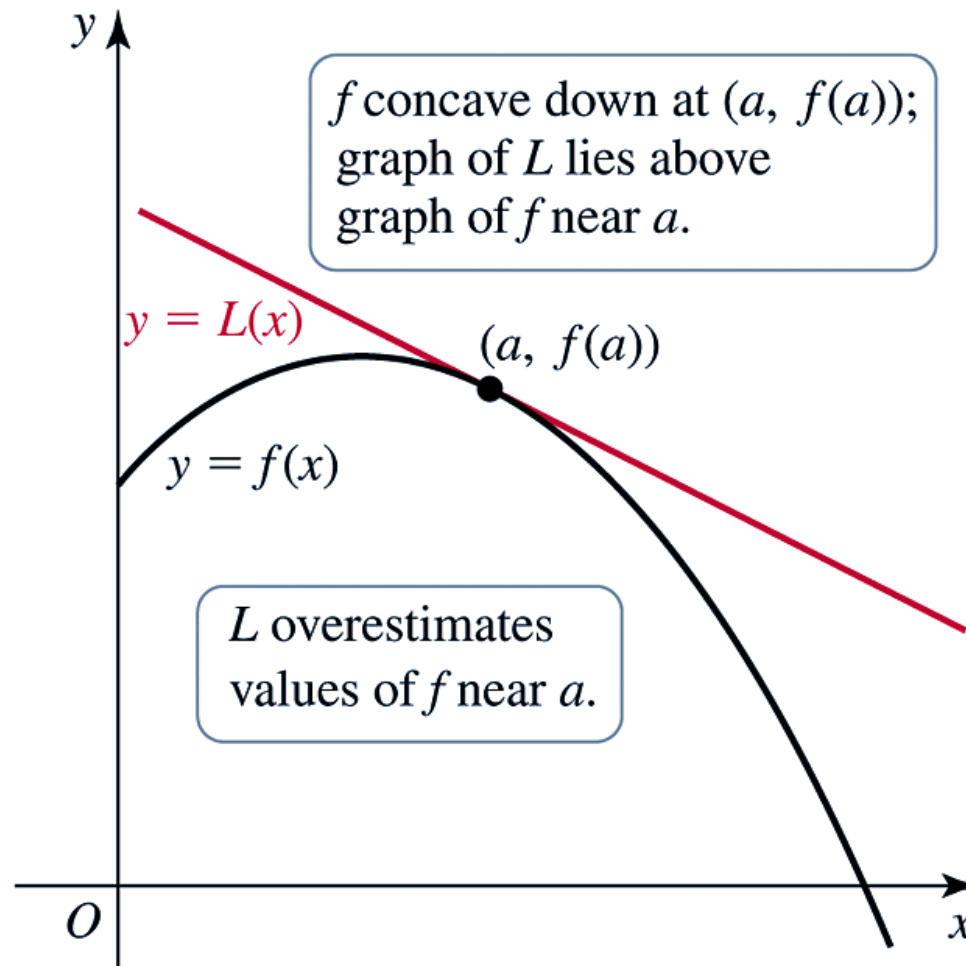


Figure 4.71 (b)



Example Estimate $\underline{2^{\frac{1}{3}}}$ and $26^{\frac{1}{3}}$.

$$\bullet f(x) = x^{\frac{1}{3}}, \quad f' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3(\sqrt[3]{x})^2}$$

$$\bullet a = 1, \quad a = 27$$

$$\bullet f(x) \approx f(1) + f'(1)(x-1) \\ = 1 + \frac{1}{3}(x-1)$$

$$\sqrt[3]{2} = f(2) \approx 1 + \frac{1}{3}$$

Figure 4.72 (a)

$$f(x) \approx f(27) + f'(27)(x-27)$$

$$= 3 + \frac{1}{3 \cdot 3^2}(x-27)$$

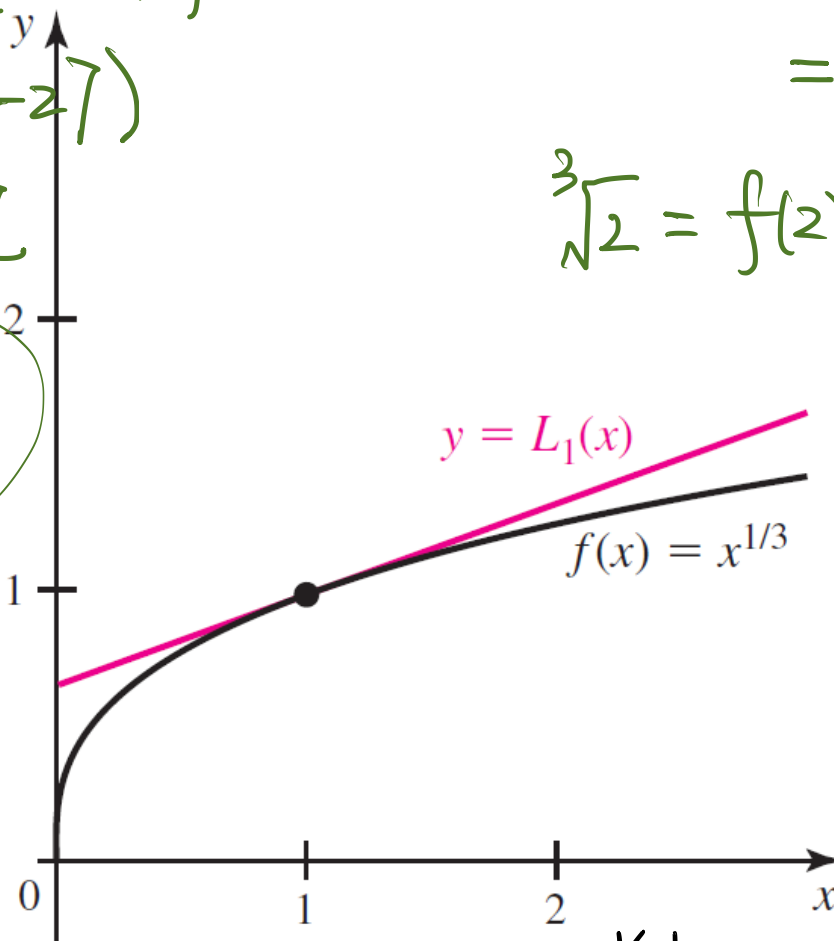
$$= 3 + \frac{x-27}{27}$$

$$\sqrt[3]{26} \approx 3 + \frac{26-27}{27}$$

$$= 3 - \frac{1}{27}$$

errors

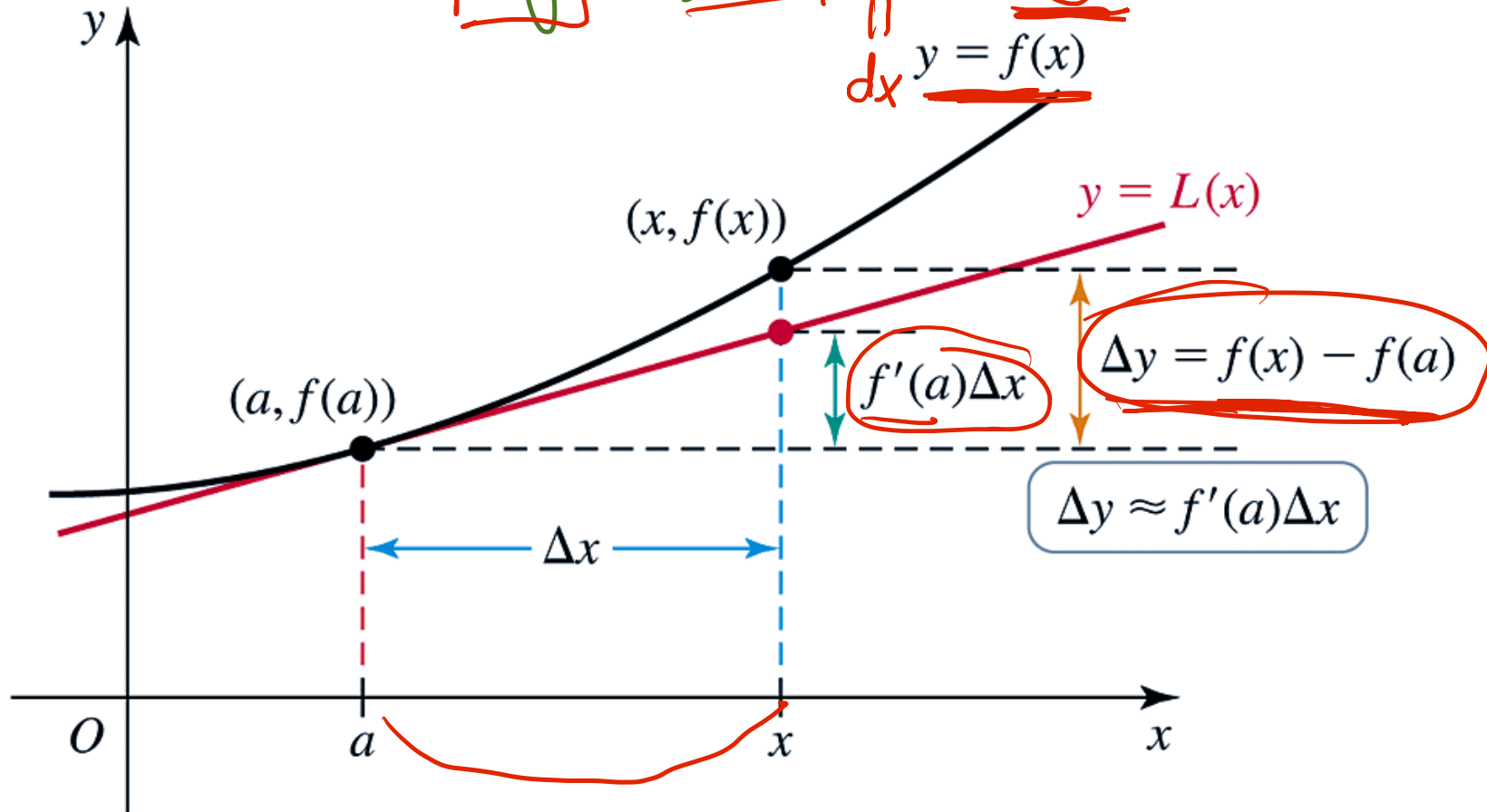
$$|L_1(2) - 2^{\frac{1}{3}}| \approx 0.073 \quad \text{and} \quad |L_2(26) - 26^{\frac{1}{3}}| \approx 0.00047$$



• Differentials $\Delta y = f(x) - f(a) \approx [f(a) + f'(a)(x-a)] - f(a)$
 $\Delta x = x - a$
 $= f'(a)(x-a)$

Figure 4.73

$\boxed{\Delta y} \approx f'(a) \boxed{\Delta x} = \underline{dy}$ $\underline{dy} = f'(a) dx$



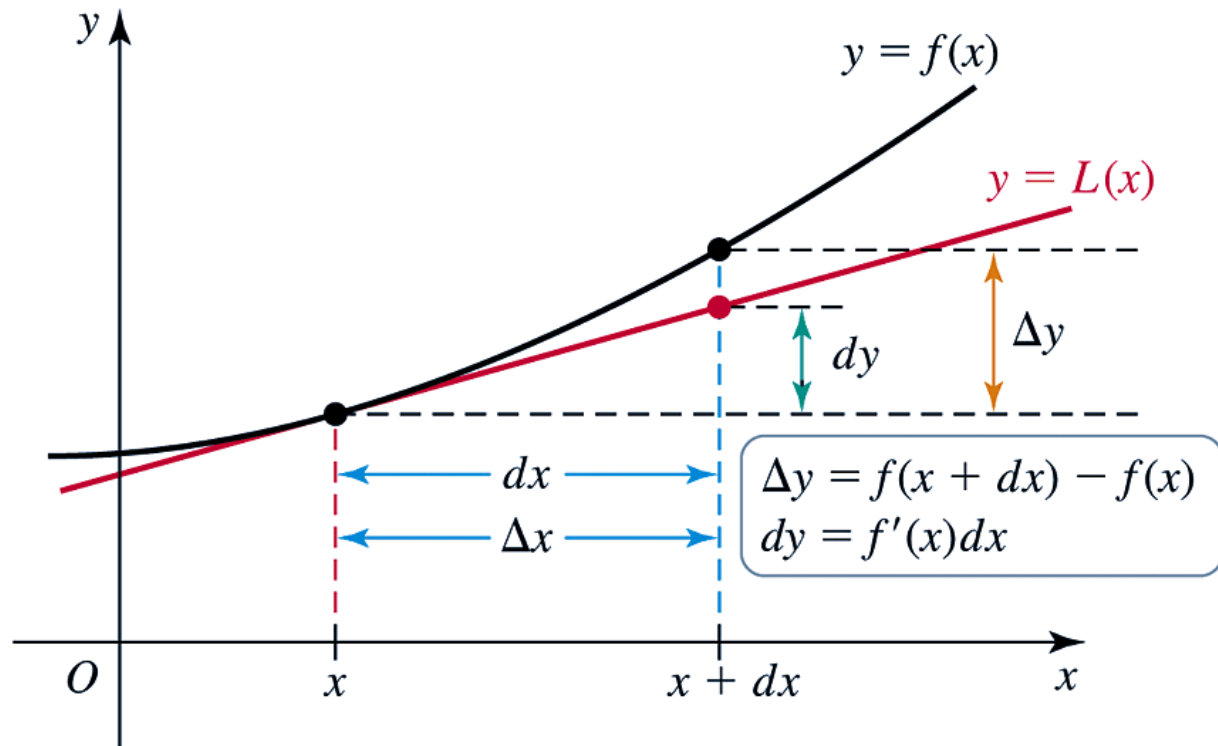
Example 6 $f(x) = 3\cos^2 x$, compute the differential $dy = f'(x) dx$

$$= 3 \cdot 2 \cos x \cdot (-\sin x) dx$$

$$= -6 \sin x \cos x dx$$

Figure 4.74

$$\frac{dy}{dx} = \frac{f'(x) dx}{dx} = f'(x).$$



Section 4.7 L'Hôpital's Rule

HW# 17, 21, 22, 31, 37, 39, 48, 69, 15, 25, 52, 53, 57, 65, 73, 79, 80, 82, 94, 99

- limits of indeterminate forms
 - $\frac{0}{0}, \frac{\infty}{\infty}$
 - $0 \cdot \infty$
 - $\frac{0}{\frac{1}{\infty}}, \frac{\infty}{\frac{1}{0}}$
 - $\frac{\infty - \infty}{\infty - \infty}$
 - $1^{\infty}, 0^0, \infty^0$
 - $\frac{\infty + \infty}{\infty + \infty}$
 - $e^{\ln 1^{\infty}} = e^{\frac{\infty \ln 1}{1}}$
- growth rates of functions.

Theorem 4.12 L'Hôpital's Rule

Suppose f and g are differentiable on an open interval I containing a with $g'(x) \neq 0$ on I when $x \neq a$.

If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm \infty$).

The rule also applies if $x \rightarrow a$ is replaced with

$x \rightarrow \pm \infty$, $x \rightarrow a^+$, or $x \rightarrow a^-$.

Theorem 4.13 L'Hôpital's Rule (Infinity over Infinity)

Suppose f and g are differentiable on an open interval I containing a , with $g'(x) \neq 0$ on I when $x \neq a$. If $\lim_{x \rightarrow a} f(x) = \pm\infty$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, then

$$\frac{\infty}{\infty}$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$).

The rule also applies for $x \rightarrow \pm\infty$, $x \rightarrow a^+$, or $x \rightarrow a^-$.

$$f(x) = x^2, \quad x^n, \quad e^x$$
$$g(x) = e^x$$

examples

Figure 4.76

$$\frac{0}{0}$$

$$= \lim_{x \rightarrow \infty} \frac{(x^2)'}{(e^x)'} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} \rightarrow \frac{\infty}{\infty}$$

$$(3) \lim_{x \rightarrow \infty} \frac{x^2}{e^x} \rightarrow \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{(2x)'}{(e^x)'} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$$

$$(4) \lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x} \rightarrow \frac{-\infty}{\infty}$$

$$= \lim_{x \rightarrow 0^+} \frac{(\ln x)'}{(\csc x)'} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\cot x \cdot \csc x}$$

$$(5) \lim_{x \rightarrow \infty} x^2 \sin\left(\frac{1}{4x^2}\right)$$

$$= \lim_{x \rightarrow \infty} \frac{\sin\left(\frac{1}{4x^2}\right)}{\frac{1}{x^2}} \rightarrow 0$$

$$= \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{4x^2}\right)' \cos\left(\frac{1}{4x^2}\right)}{\left(\frac{1}{x^2}\right)'} = \lim_{x \rightarrow \infty} \frac{\frac{1}{4} \cos\left(\frac{1}{4x^2}\right)}{\frac{1}{x^2}} = \frac{1}{4}$$

$$(1) \lim_{x \rightarrow 0} \frac{\sin x}{x} \rightarrow \frac{0}{0}$$

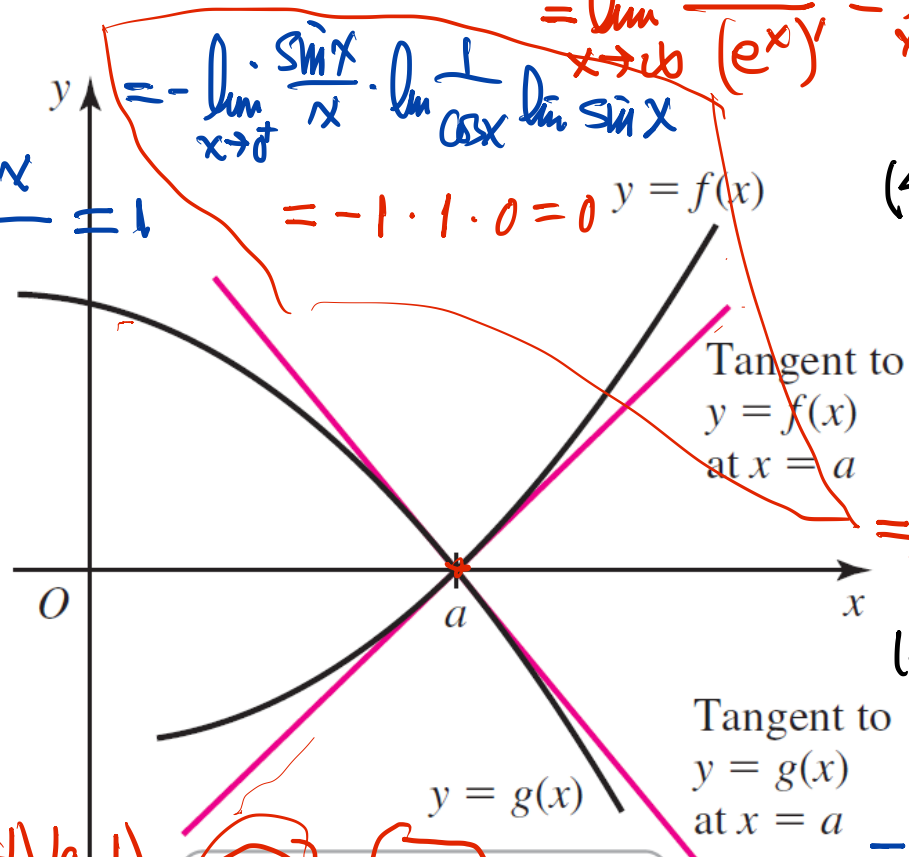
$$= \lim_{x \rightarrow 0} \frac{(\sin x)'}{(x)'} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$$

$$(2) \lim_{x \rightarrow 0} \frac{e^x - x - 1}{x} \rightarrow \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{e^x - 1}{1} = 0$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$(6) \lim_{x \rightarrow \infty} \left[x - \sqrt{x^2 - 3x} \right] \frac{[x + \sqrt{x^2 - 3x}]}{x + \sqrt{x^2 - 3x}}$$



$$\frac{f(x)}{g(x)} \rightarrow \frac{f'(x)}{g'(x)} \text{ as } x \rightarrow a$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 - 3x) = 3x}{x + \sqrt{x^2 - 3x}} = \lim_{x \rightarrow \infty} \frac{3x}{x + \sqrt{x^2 - 3x}} = \lim_{x \rightarrow \infty} \frac{3}{1 + \sqrt{1 - 3/x}} = \frac{3}{2}$$

Procedure Indeterminate Forms 1 to the Power Infinity, 0 to the Power 0, and Infinity to the Power 0

Assume $\lim_{x \rightarrow a} f(x)^{g(x)}$ has the indeterminate form 1^∞ , 0^0 , or ∞^0 .

1. Analyze $L = \lim_{x \rightarrow a} g(x) \ln f(x)$. This limit can be

put in the form $0/0$ or ∞/∞ ,

both of which are handled by l'Hôpital's Rule.

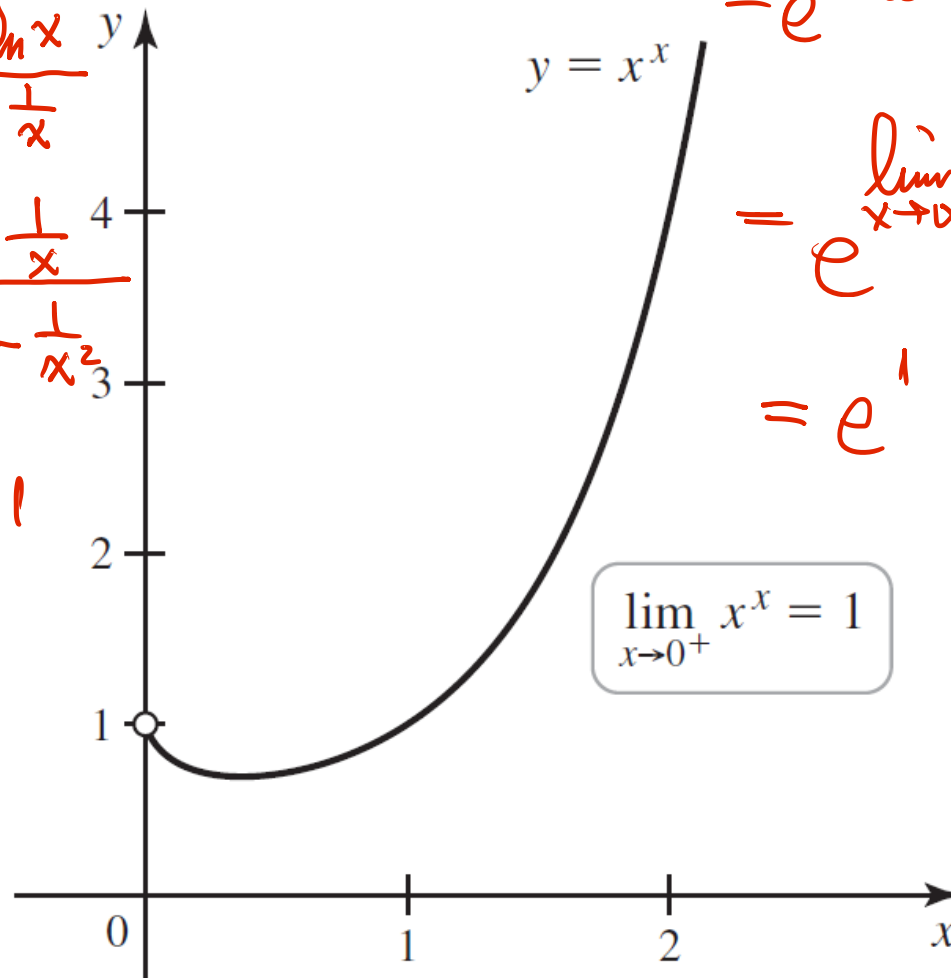
2. When L is finite, $\lim_{x \rightarrow a} f(x)^{g(x)} = e^L$. if $L = \infty$ or $-\infty$, then

$\lim_{x \rightarrow a} f(x)^{g(x)} = \infty$ or $\lim_{x \rightarrow a} f(x)^{g(x)} = 0$, respectively.

$$f^g = e^{\ln f^g} = e^{\underline{g \ln f}}$$

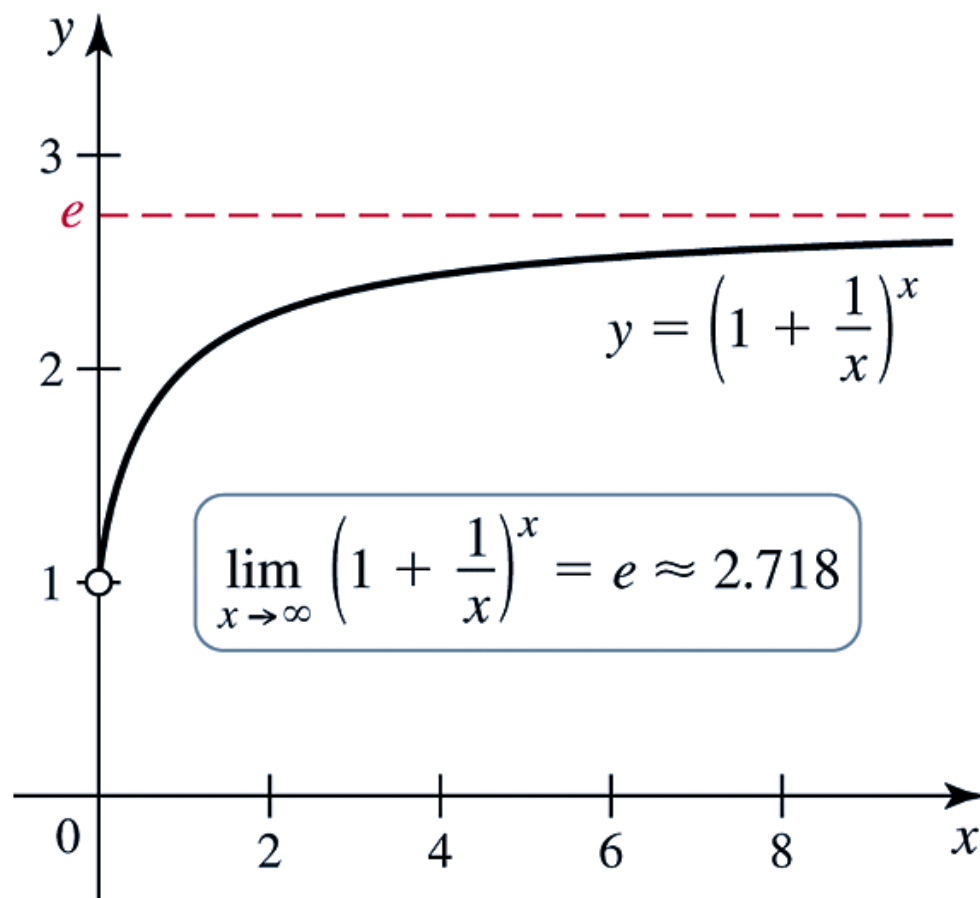
Figure 4.77

$$\begin{aligned} (7) \lim_{x \rightarrow 0^+} x^x &= 0^0 \\ &= \lim_{x \rightarrow 0^+} e^{x \ln x} = e^{\lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}} \\ &= e^{\lim_{x \rightarrow 0^+} \frac{(\ln x)'}{(\frac{1}{x})'}} = e^{\lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}} \\ &= e^{\lim_{x \rightarrow 0^+} (-x)} = e^0 = 1 \end{aligned}$$



$$\begin{aligned} (8) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x &= \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \\ &= e^{\lim_{x \rightarrow \infty} x \ln \left(1 + \frac{1}{x}\right)} = e^{\lim_{x \rightarrow \infty} \frac{[\ln(1 + \frac{1}{x})]'}{(\frac{1}{x})'}} \\ &= e^{\lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{1}{x}} \cdot (-\frac{1}{x^2})}{(-\frac{1}{x^2})}} \\ &= e^1 = e \end{aligned}$$

Figure 4.78



Definition Growth Rates of Functions as x Approaches Infinity

Suppose f and g are functions with

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty.$$

Then **f grows faster than g** as $x \rightarrow \infty$ if

$$\lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = 0 \quad \text{or, equivalently, if} \quad \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty.$$

The functions f and g have **comparable growth rates** if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = M,$$

where $0 < M < \infty$ (M is nonzero and finite).

Figure 4.79

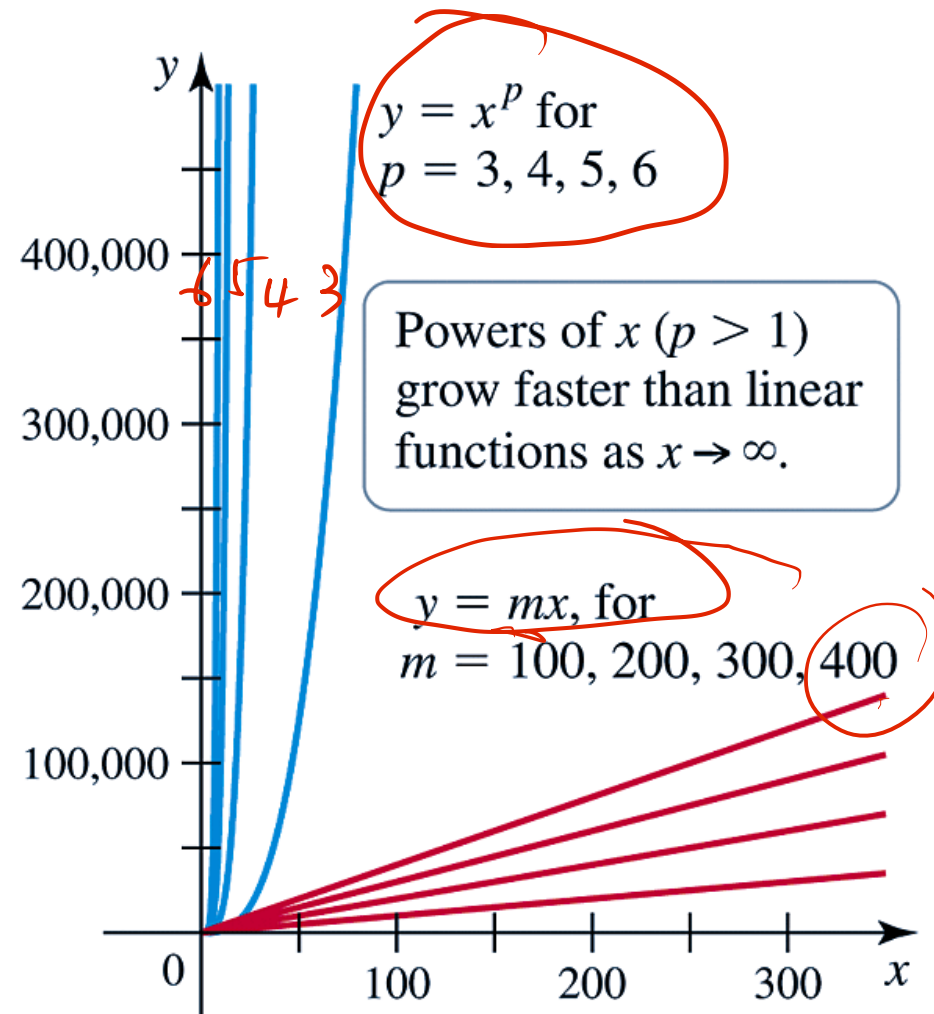
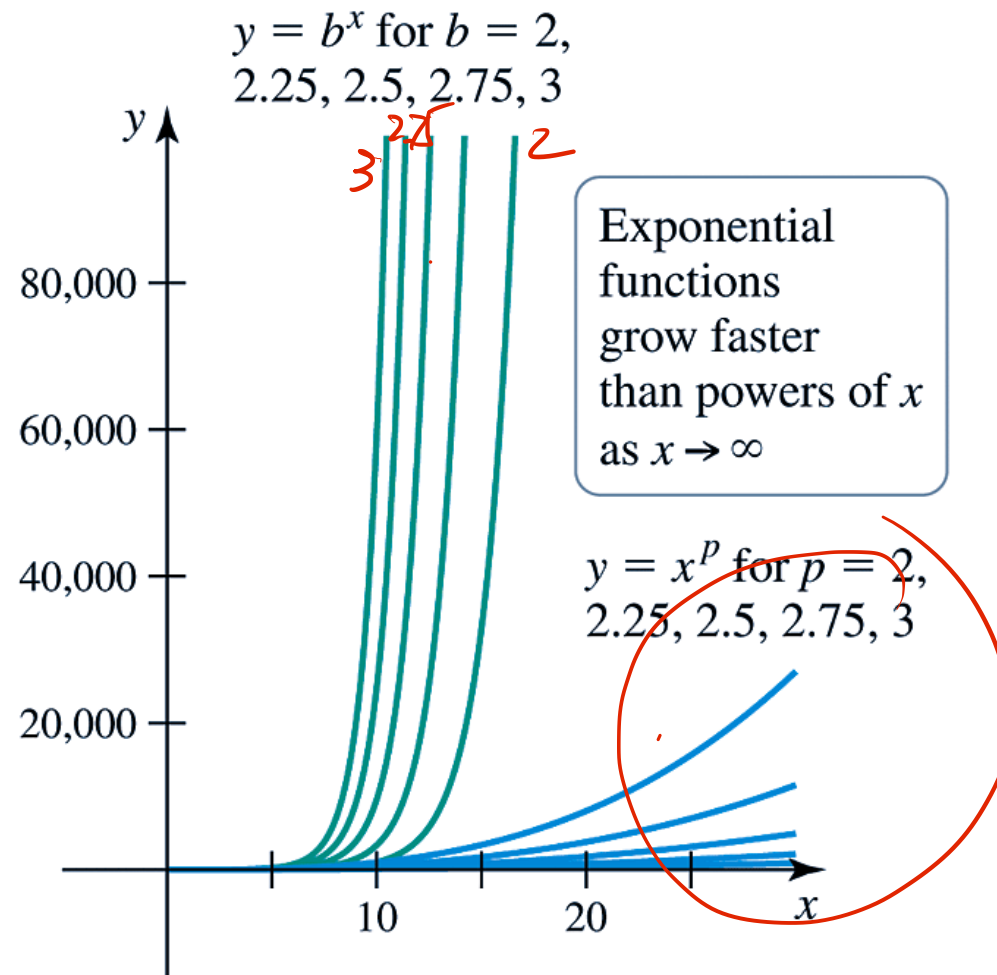


Figure 4.80



Theorem 4.14 Ranking Growth Rates as x Approaches Infinity

Let $f \ll g$ mean that g grows faster than f as $x \rightarrow \infty$. With positive real numbers p , q , r , and s and with $b > 1$,

$$\ln^q x \ll x^p \ll x^p \ln^r x \ll x^{p+s} \ll b^x \ll x^x.$$

Section 4.9 Antiderivatives

HW # 4, 13, 17, 21, 25, 27, 34, 35, 40, 43, 62, 69, 73, 79, 86, 91, 97, 107, 111, 112.

Definition Antiderivative

A function F is an **antiderivative** of f on an interval I .

provided $F'(x) = f(x)$, for all x in I .

- Theorem F is an antiderivative of $f \implies$ so is $F + C$ for any constant C .
$$\underline{F' = f} \implies F' + C' = (F + C)' = f$$

- Indefinite Integral

$$\int f dx = F + C$$

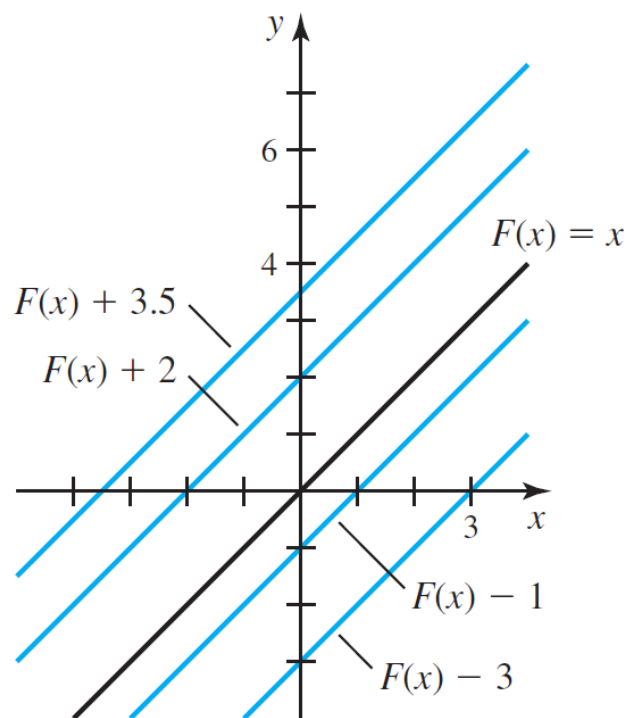
examples

$$\text{ii) } \int 1 dx = x + C$$

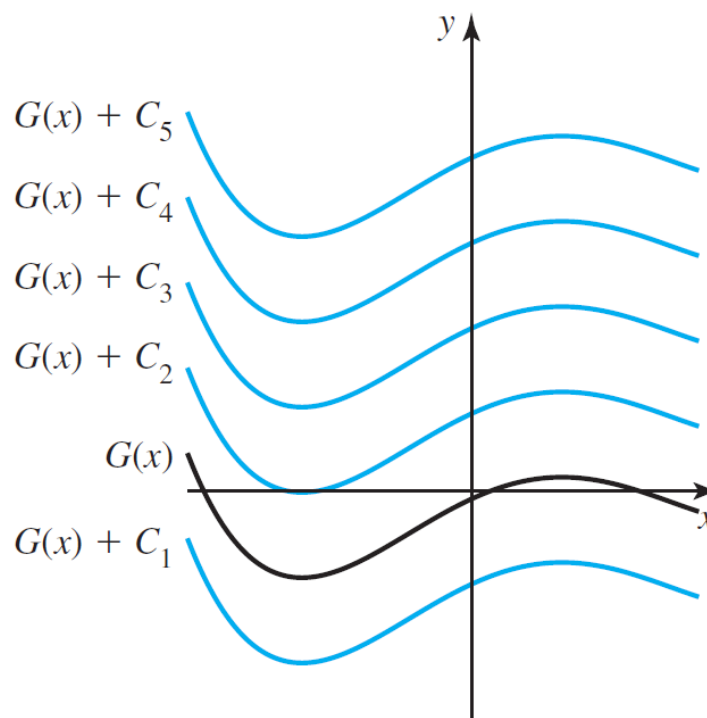
$$G'(x) = g(x)$$

$$\Rightarrow \int g(x) dx = G(x) + \underline{\underline{C}}$$

Figure 4.90



Several antiderivatives of $f(x) = 1$ from the family $F(x) + C = x + C$



If G is any antiderivative of g , the graphs of the antiderivatives $G + C$ are vertical translations of one another.

Theorem 4.17 Constant Multiple and Sum Rules

Constant Multiple Rule: $\int c f(x) dx = c \int f(x) dx$, for real numbers c

Sum Rule: $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$

$$(x^p)' = p x^{p-1}$$

Power Rule ($p \neq -1$): $\int x^p dx = \frac{1}{p+1} x^{p+1} + C$

$$(1) \int 3x^2 dx = 3 \cdot \frac{1}{3} x^3 = x^3 + C$$

$$(2) \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$(3) \int \sin t dt = -\cos t + C$$

Table 4.9

-
1. $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x \, dx = \sin x + C$
 2. $\frac{d}{dx}(\cos x) = -\sin x \Rightarrow \int \sin x \, dx = -\cos x + C$
 3. $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x \, dx = \tan x + C$
 4. $\frac{d}{dx}(\cot x) = -\csc^2 x \Rightarrow \int \csc^2 x \, dx = -\cot x + C$
 5. $\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x \, dx = \sec x + C$
 6. $\frac{d}{dx}(\csc x) = -\csc x \cot x \Rightarrow \int \csc x \cot x \, dx = -\csc x + C$
-

Table 4.10

$$7. \quad \frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + C$$

$$8. \quad \frac{d}{dx}(\ln |x|) = \frac{1}{x} \Rightarrow \int \frac{dx}{x} = \ln |x| + C$$

$$9. \quad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$10. \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \Rightarrow \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$11. \quad \frac{d}{dx}(\sec^{-1}|x|) = \frac{1}{x\sqrt{x^2-1}} \Rightarrow \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x| + C$$

$$\int (3x^5 + 2 - 5\sqrt{x}) dx = 3 \int x^5 dx + 2 \int 1 dx - 5 \int \sqrt{x} dx$$

$\parallel$
 $x^{\frac{1}{2}}$

$$= 3 \cdot \frac{1}{6} x^6 + 2x - 5 \cdot \frac{2}{3} x^{\frac{3}{2}} + C$$

$$= \frac{1}{2} x^6 + 2x - \frac{10}{3} x^{\frac{3}{2}} + C$$

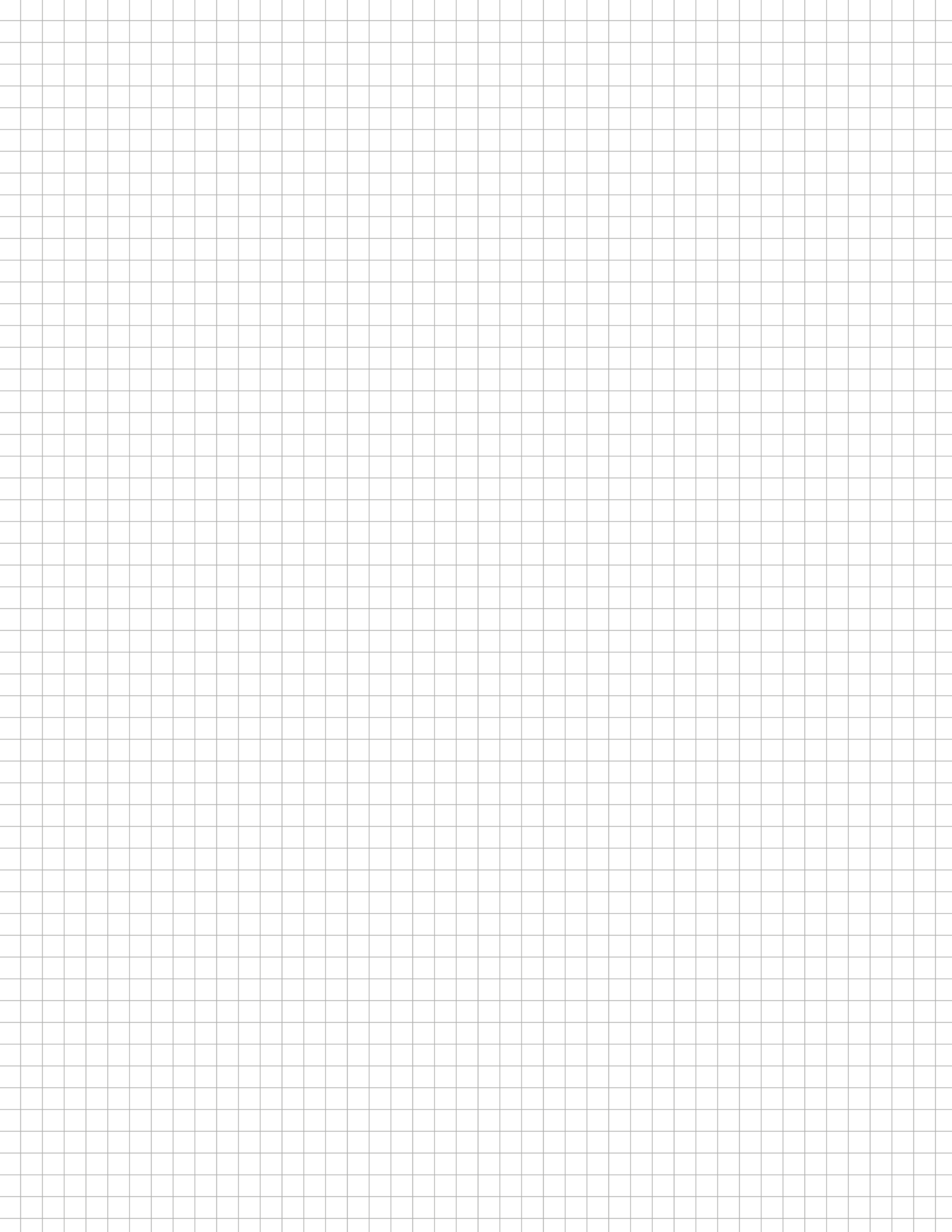
$$\int \frac{x^2 - 3}{x^2 + 1} dx = \int \frac{x^2 + 1 - 4}{x^2 + 1} dx = \int \left[1 - 4 \cdot \frac{1}{x^2 + 1} \right] dx$$

$$= x - 4 \tan^{-1} x + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int \left(\frac{4}{3\sqrt{1-x^2}} - \frac{3}{x} \right) dx = \frac{4}{3} \int \frac{1}{\sqrt{1-x^2}} dx - 3 \int \frac{1}{x} dx$$

$$= \frac{4}{3} \arcsin x - 3 \ln|x| + C$$



- Initial Value Problem

$$\begin{cases} f'(x) = x^2 - 2x \\ f(1) = \frac{1}{3} \end{cases}$$

$f(x) = ?$
ordinary diff. eq.

Figure 4.91

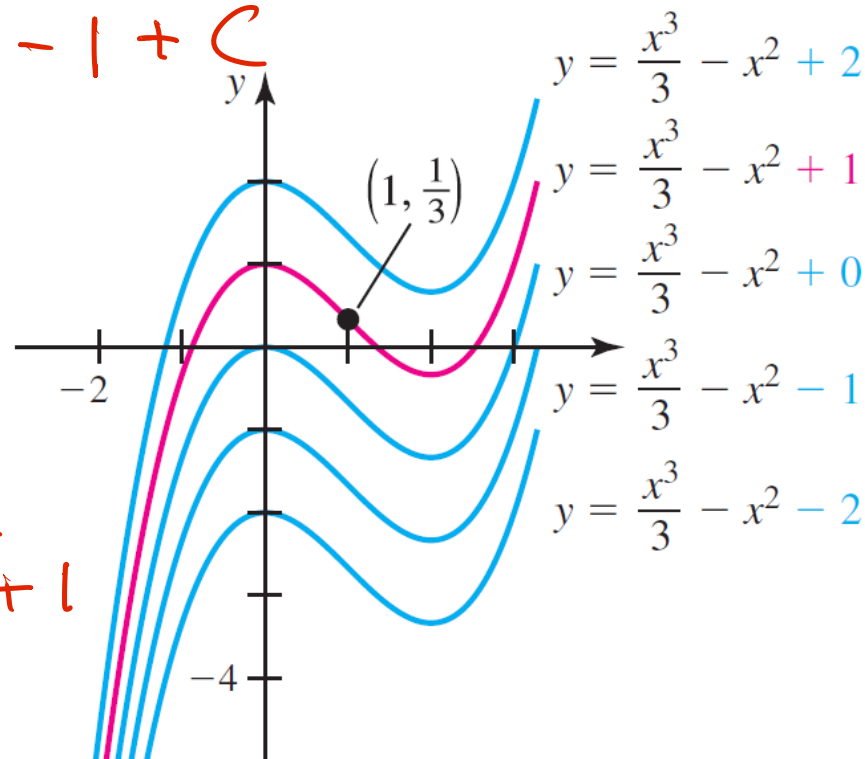
$$\Rightarrow \underline{f(x)} = \int f' dx = \int (x^2 - 2x) dx = \underline{\underline{\frac{1}{3}x^3 - x^2 + C}}$$

Antiderivatives
of $x^2 - 2x$.

$$\begin{aligned} \frac{1}{3} &= f(1) = \frac{1}{3} \cdot 1^3 - 1^2 + C \\ &= \frac{1}{3} - 1 + C \end{aligned}$$

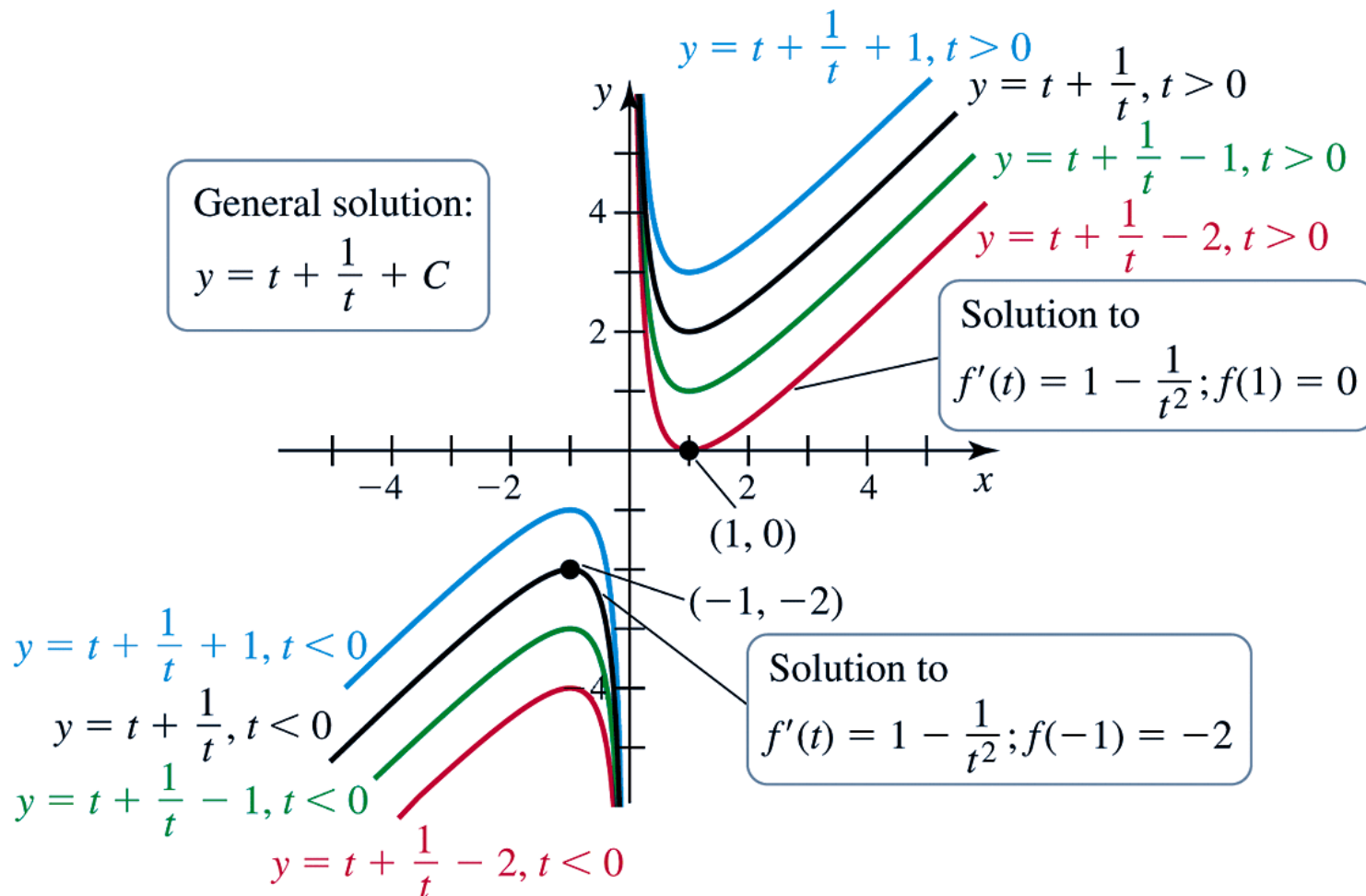
$$\Rightarrow C = 1$$

$$\Rightarrow f(x) = \frac{1}{3}x^3 - x^2 + 1$$



$$\begin{cases} f'(t) = 1 - \frac{1}{t^2} \\ f(1) = 0 \end{cases}$$

Figure 4.92



Example 9 A stone is thrown vertically upward at $t=0$ with a velocity of 40 m/s from the edge of a cliff that is 100 m above a river.

- (a) Find the velocity $v(t)$ of the object for $t \geq 0$. (b) Find the position $s(t)$.

Figure 4.94

- (c) the maximum height above the river.

- (d) With what speed does the object strike the river?

Given

$$v(0) = 40 \text{ m/s}$$

$$s(0) = 100 \text{ m}$$

$$a(t) = -9.8 \text{ m/s}^2$$

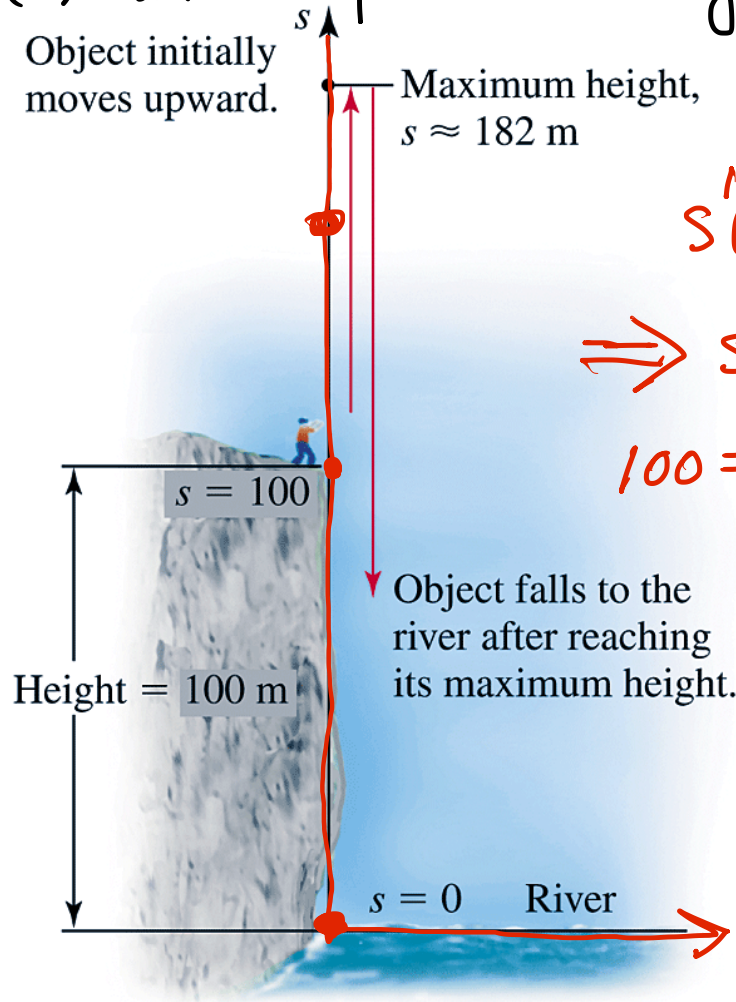
||

$$\int v'(t) = -9.8$$

$$\begin{cases} v(0) = 40 \text{ m/s} \end{cases}$$

$$v(t) = -9.8t + C$$

$$40 = v(0) = C$$



$$v(t) = -9.8t + 40$$

$$s'(t) = v(t) = -9.8t + 40$$

$$\Rightarrow s(t) = -4.9t^2 + 40t + C$$

$$100 = s(0) = C$$

$$s(t) = -4.9t^2 + 40t + 100$$

Figure 4.95

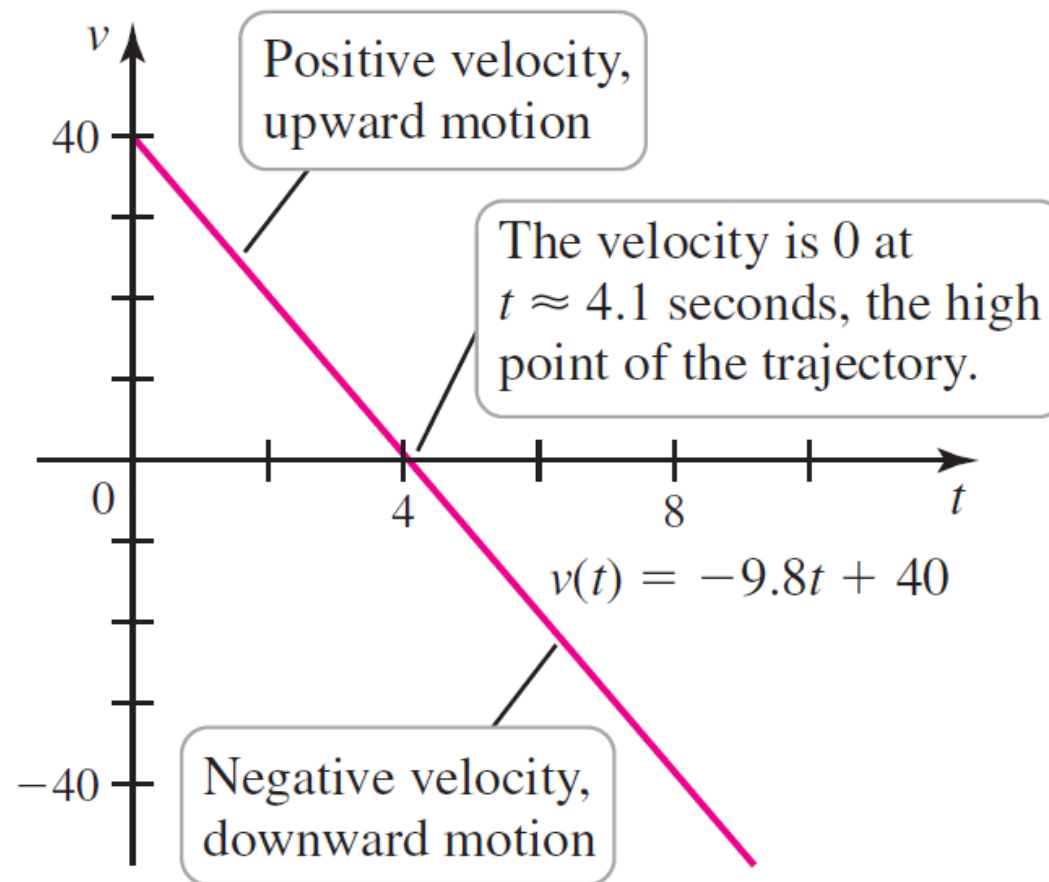


Figure 4.96

